AN INTRODUCTION TO

ANALYTICAL PLANE GEOMETRY.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

AN INTRODUCTION

TO

ANALYTICAL PLANE GEOMETRY

BY

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Cambridge:
DEIGHTON, BELL, AND CO.
LONDON: BELL AND DALDY.
1867

PREFACE.

In the earlier part of this book—the part in which the method of co-ordinates is described and applied to the straight line and the conic sections—I have adopted the plan of Mr Todhunter's *Co-ordinate Geometry*, examining the parabola, ellipse, and hyperbola before discussing the general equation of the second degree.

In the thirteenth chapter some account is given of projection and reciprocal polars. In applying perspective projection I have not ventured beyond real conceivable geometry.

The last chapter introduces equations of higher degrees than the second.

The reader of Mr Salmon's works will anticipate my great obligations to the *Conic Sections* and *Higher Plane Curves*. I would also make grateful mention of Mr Ferrers' *Trilinear Co-ordinates*, though I have not studied to keep strictly apart the methods of trilinear co-ordinates and abridged notation.

W. P. TURNBULL.

TRINITY COLLEGE,
May, 1867.

CONTENTS.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

CHAPTER XIV.

Readers who are studying this subject for the first time may find it good to omit most of Chapters IV, V, VII, XI, XII, XIII, XIV, and to read no chapter completely before trying its examples.

ERRATA.

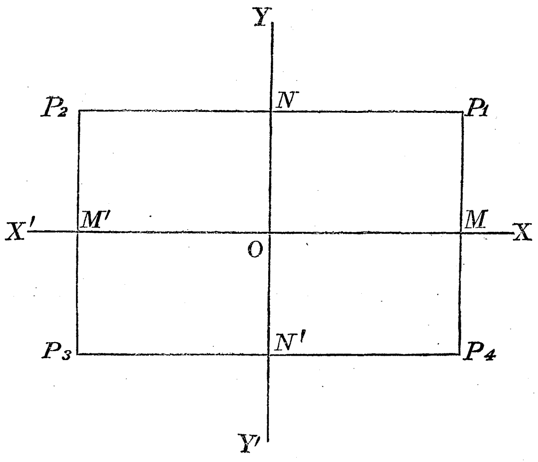
Page 2, line 15. For X' read XX'.
,, — ,, 16. For OX read OY.
,, 21, ,, 1. For Ox read OX.
,, 22, ,, 4. For ba read BA.
,, 38, Equations (2), (3) should be $Ar_1 \cos\theta_1 + Br_1 \sin\theta_1 + C = 0$,
$Ar_2 \cos\theta_2 + Br_2 \sin\theta_2 + C = 0$.
,, 41, line 1. For AB read AC.
,, 44, Ex. 2. For 20 read 17.
,, 47, Ex. 39. For 'reciprocal' read 'reciprocals'.
,, 51, line 5 of Art. 56. For 'lines' read 'line'.
,, 57, Ex. 1 (4). For $4xy^2$ read $4xy$.
,, — ,, (5). For $4y^2$ read $4x^2$.
,, 59, Ex. 20. For 'a rigid' read 'an equilateral'.
,, 68, line 9. Strike out 'proportional to'. Further on, for 'the form (1)' read 'a form like (1), determining l, m, n only as regards their ratios'.
,, 73, Ex. 10. For 'line' read 'lines'.
,, — ,, 13. For c read l.
,, 76, Equation (2). For $-2(a+b\cos\omega)y$ read $-2(b+a\cos\omega)y$.
,, 81, line 9. For k^2 read c^2.
,, 83, ,, 3 of Art. 91. For y read x.
,, 86, Ex. 8. For 'relation between A, B, and C' read 'relations between A, B, and C necessary'.
,, 89, Ex. 32. For x^2 read x.
,, 97, Ex. 7. For 'curve' read 'curves'.
,, 98, line 6. For L, N, D read L, M, N. For 'polar' read 'pole'; and strike out the reference.
,, 109, Ex. 10. For $(-2, 4)$ read $(4, -2)$.
,, — Ex. 11. For $\sqrt{3}$ read 3.
,, 111, Ex. 29. For 'points' read 'point'.
,, 130, Ex. 36. For 'sums...of central...are' read 'sum...of two central...is'.
,, — Ex. 39. After 'the length of' insert 'a'.
,, 142, Art. 159. For 'the two axes' read 'the two co-ordinate axes'.
,, 151, Ex. 41. For $3a^2-b^2$ and $3b^2-a^2$ read a^2-3b^2 and b^2-3a^2.
,, — Ex. 44. For 'those' read 'any'.
,, 167, Equation (2). For $2c'$ read $2c'xy$.
,, 192, line 2. Strike out the comma after '(Art. 194)'.
,, 231, line 2. For $\sqrt{2}$ read $\sqrt{3}$.
,, 232, Ex. 22. For 'Ex. 30 and...into propositions' read 'Ex. 30 or ...into a proposition', and for 'Deduce from the circle the' read 'Deduce a property of the circle from the'.
,, 256, Chapter VI. Ex. 4. For 'if $a > \pi$ and < 0' read 'if $a > 0$ and $< \pi$'.

CHAPTER I.

INTRODUCTION.

1. ALGEBRA enters into Geometry in two ways. It may be applied, as in Trigonometry, to problems concerning the *magnitudes* of lines, areas, volumes, or angles; and it may be applied to problems concerning the *positions* of lines and surfaces. Both these applications are made in the subject of Analytical Geometry, which in its most general form deals with space of three dimensions; and both are made in that special and simpler branch of Analytical Geometry which is concerned only with figures lying in one plane.

2. Let there be chosen in the plane of operations a point O



and two unlimited straight lines $X'OX$, $Y'OY$ passing through O, and intersecting at right angles. Then suppose we know that a certain point in the plane is at a distance b from the line $X'OX$, and at a distance a from the line $Y'OY$, and that we are required to find this point. Take M, M' in XX', such that $OM = OM' = a$, and N, N' in YY', such that $ON = ON' = b$, and draw through M, M' parallels to XX', and through N, N' parallels to YY', so as to form the rectangle $P_1P_2P_3P_4$. Then we know that the point in question is one of the four angular points of this rectangle; and we know no more about the position of the point. But suppose that distances from YY' are positive or negative according as they are on the OX-side or the OX'-side of YY', and that the sign $+$ is placed before a; then we know that the point in question is either P_1 or P_2. Again, suppose also that distances from X' are positive or negative according as they are on the OY-side or the OX- side of XX', and that the sign $+$ is placed before b; then we know that the point in question is P_1.

If for $+a$, $+b$ there had been given $-a$, $+b$, the point in question would have been P_2; if $-a$, $-b$, P_3; if $+a$, $-b$, P_4.

3. The distances (properly signed) of a point from the fixed lines XX', YY' are called the *co-ordinates* of the point. The co-ordinate measured parallel to XX' is usually denoted by the symbol x, and that measured parallel to YY' by the symbol y. Thus the x of the point P_1 is $+a$, and the y is $+b$; or, more concisely, for the point P_1, $x = a$, $y = b$. Similarly for the point P_2, $x = -a$, $y = b$; for the point P_3, $x = -a$, $y = -b$; for the point P_4, $x = a$, $y = -b$.

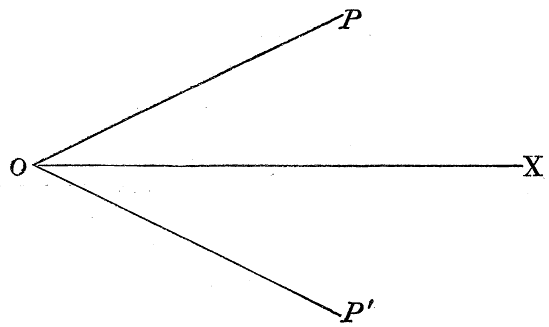
We may also speak of the point P_1 as the point (a, b), and of the points P_2, P_3, P_4 as the points $(-a, b)$, $(-a, -b)$, $(a, -b)$ respectively. Or we may, when there is no risk of confusion, omit the brackets, and, in the case of P_1, the comma: thus P_1 is the point ab.

4. If instead of choosing XX' and YY' *at right angles* we choose them inclined at *any* angle, and if we measure parallel to XX' distances from YY', and parallel to YY' distances

from XX', and alter the word 'rectangle' to the more general term 'parallelogram,' then Arts. 2, 3 may be repeated.

The point O is called the *origin of co-ordinates*, or the *origin.* The lines XX', YY' are called the *co-ordinate axes*, or the *axes:* XX' is the *axis of x* and YY' is the *axis of y.* When the angle of XOY is a right angle the axes and the co-ordinates are said to be *rectangular;* when the angle XOY is not a right angle the axes and the co-ordinates are said to be *oblique.* It is commonly understood, when no intimation is given to the contrary, that the axes are rectangular.

5. Another apparatus for registering the position of a point in the plane of operations consists simply of a fixed point and a fixed line drawn *from* this point. The method which employs this apparatus is called the method of *polar co-ordinates.* Let O



be the fixed point and OX the fixed line; and suppose we know that a certain point in the plane is at a distance a from O and that the line joining it to O makes an angle α with OX, and that we are required to find this point.

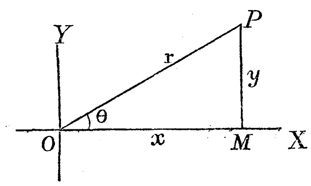
Make the angles XOP, XOP', on opposite sides of OX, each equal to α, and make OP, OP' each equal to a. Then we know that the point in question is either P or P'. There will be no doubt *which* of these two points is to be taken, if angular distances measured from OX be considered positive or negative according as they are measured towards OP or OP', and if a sign be prefixed to α.

6. The point O is called the *origin* or *pole*, and the line OX the *initial line*: OP is the *radius vector* of the point P, and XOP the *vectorial angle* of the point P. Let the *general* symbols for the radius vector and the vectorial angle be respectively r and θ. Then the polar co-ordinates of the point P are a, α; or, for the point P, $r = a$, $\theta = \alpha$; or P is the point (a, α). For the point P', $r = a$, $\theta = -\alpha$; or, P' is the point $(a, -\alpha)$.

7. The angle α has been supposed $< \pi$, but by the Trigonometrical extension of the term 'angle' we may express the position of *any* point without a negative vectorial angle. Thus in the figure P' is the point $(a, 2\pi - \alpha)$. In like manner P is the point $(a, -\overline{2\pi - \alpha})$.

Again, the radius vector also is capable of two signs. Produce PO to P_1, so that $OP_1 = OP$. Then the symbol $(-a, \alpha)$ may be used to express the point P_1. If we wish to express P_1 with a *positive* radius vector, we may call it the point $(a, \overline{\pi + \alpha})$ or the point $(a, -\overline{\pi - \alpha})$. So P is the point $(-a, \pi + \alpha)$. The points (r, θ), $(-r, \theta)$ are on a straight line through the pole and are equidistant from the pole.

8. Let x, y be the co-ordinates of any point P referred to the rectangular axes OX, OY, and r, θ the polar co-ordinates of



the same point referred to O and OX, vectorial angles being considered positive when measured from OX towards OY. Then, PM being perpendicular to OX, we see from the right-angled triangle OPM that

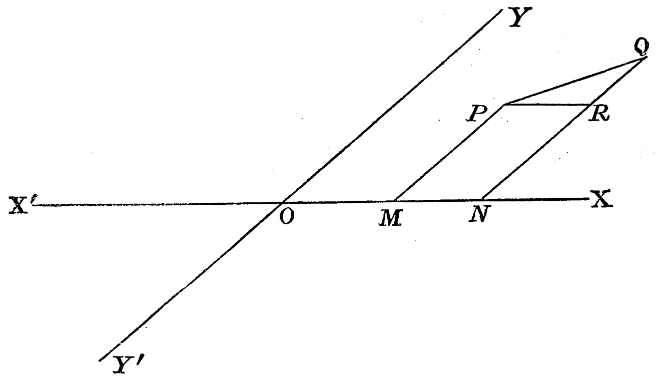
$$x = r\cos\theta, \text{ and } y = r\sin\theta \quad\ldots\ldots\ldots\ldots(1);$$

and that $$r^2 = x^2 + y^2, \text{ and } \tan\theta = \frac{y}{x} \quad\ldots\ldots\ldots\ldots(2).$$

Thus if r and θ be known, x and y are known from (1), and if x and y be given, we can find r and θ from (2).

9. We now proceed to express analytically the length of the line joining two given points $(x_1 y_1)$, $(x_2 y_2)$.

The axes are supposed to be inclined at an angle ω.



Let P be the point $x_1 y_1$ and Q the point $x_2 y_2$.

Draw PM, QN parallel to OY and PR parallel to OX, so that

$$x_1 = OM,\ y_1 = PM,\ x_2 = ON,\ y_2 = QN.$$

Then $$PR = x_2 - x_1,\ QR = y_2 - y_1,$$

and $$PQ^2 = PR^2 + QR^2 - 2PR \,.\, QR \cos PRQ$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos \omega \ldots\ldots (1).$$

In the particular case of rectangular axes $\omega = \frac{\pi}{2}$, and

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Suppose we require an expression for the distance of a point xy from the origin, the axes being inclined at an angle ω.

We may write x, y for x_2, y_2, in (1), and 0, 0 (the coordinates of the origin) for x_1, y_1. Hence we find that the square of the distance is

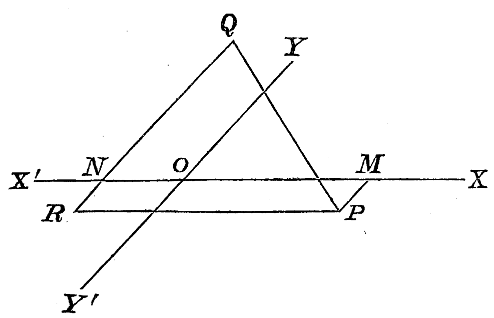
$$x^2 + y^2 + 2xy \cos \omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

as might also have been proved independently by a figure.

10. In obtaining the formula (1) the quantities x_1, y_1, x_2, y_2 were supposed positive, so that the points P, Q were both in the *first compartment*, as the space within the angle XOY is called. But the formula will give accurately the distance between *any* two points whose co-ordinates are known. Suppose for instance that x_1, y_2 are positive and y_1, x_2 negative, so that P lies in the *fourth* compartment (or within the angle XOY'), and Q in the *second* compartment (or within the angle $X'OY$). Construct the figure as in Art. 9. Then

$$PQ^2 = PR^2 + QR^2 - 2PR \cdot QR \cos\omega,$$

for the angle $$PRQ = \omega.$$



Now $\quad PR = MN = OM + ON = x_1 + (-x_2),$

(for, since x_2 is negative, the *numerical* measure of ON is $-x_2$) $= x_1 - x_2$.

And $\quad QR = QN + NR = y_2 + (-y_1) = y_2 - y_1;$

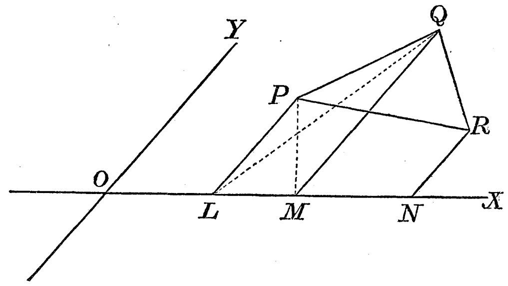
$$\therefore PQ^2 = (x_1 - x_2)^2 + (y_2 - y_1)^2 - 2(x_1 - x_2)(y_2 - y_1)\cos\omega,$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega.$$

And any other case may be treated in like manner.

11. *To express the area of a triangle in terms of the co-ordinates of the angular points.*

Let P, Q, R, the angular points of the triangle, be x_1y_1,

x_2y_2, x_3y_3 respectively, the axes being inclined at an angle ω.



Draw the ordinates (as they are called) PL, QM, RN parallel to OY.

The area of any triangle is measured by half the product of two sides and the sine of the included angle.

The triangle PQR

$$= \text{figure } PLMQ + \text{figure } QMNR - \text{figure } RNLP.$$

Join PM, QL. The figure $PLMQ$

$$= \text{triangle } PLM + \text{triangle } PMQ,$$
$$= \text{triangle } PLM + \text{triangle } QML, \text{ (Euclid, I. 37)}$$
$$= \tfrac{1}{2}(PL + QM)\,ML \sin\omega,$$
$$= \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1) \sin\omega.$$

Similarly, the figure QN

$$= \tfrac{1}{2}(y_2 + y_3)(x_3 - x_2) \sin\omega,$$

and the figure PN

$$= -\tfrac{1}{2}(y_3 + y_1)(x_1 - x_3) \sin\omega.$$

Thus the area of the triangle is

$$\frac{\sin\omega}{2}\{(y_2+y_3)(x_3-x_2)+(y_3+y_1)(x_1-x_3)+(y_1+y_2)(x_2-x_1)\},$$

which expression may be reduced to

$$\frac{\sin\omega}{2}\{x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1 + x_2y_1 - x_1y_2\} \ldots\ldots\ldots (1).$$

In the particular case of rectangular axes the area of the triangle is

$$\tfrac{1}{2}\{x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1 + x_2y_1 - x_1y_2\} \text{..........}(2).$$

If, in any particular case, the formula for the area of a triangle give a quantity with a negative sign, this sign must be changed. We may if we please change the sign of the expression for the area, and use instead of (1) the formula

$$\frac{\sin\omega}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \text{..........}(3).$$

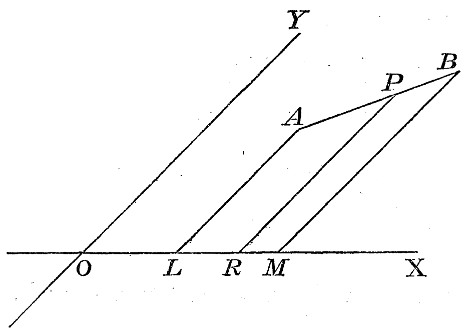
The reader will observe that the term $x_2(y_3 - y_1)$ can be obtained from $x_1(y_2 - y_3)$ by changing 1, 2, 3 to 2, 3, 1. The same change derives $x_3(y_1 - y_2)$ from $x_2(y_3 - y_1)$ and $x_1(y_2 - y_3)$ from $x_3(y_1 - y_2)$. The suffix 3 is to be considered as coming just before the suffix 1. Thus 123, 231, 312 are, in a manner, the same order of suffixes, and in like manner 321, 213, 132 are all one order, which is the opposite of the former.

Ex. The area of the triangle formed by joining the points x_1y_1, x_2y_2 to each other and to the origin is

$$\pm \tfrac{1}{2}\sin\omega\,(x_2y_1 - x_1y_2).$$

The area in (1) is expressed as an algebraic sum of three such areas.

12. *To find the co-ordinates of the point which divides in a given ratio the line joining two given points.*



Let xy be the point P which divides in the given ratio $n_1 : n_2$, the line joining the given point x_1y_1, or A, to a given point x_2y_2, or B. Then $AP : BP = n_1 : n_2$. (BA is divided in the ratio $n_2 : n_1$). Draw the ordinates AL, BM, PR. Then since (Euclid, Book VI.) all straight lines which are cut by a system of parallels are cut in the same proportion,

$$\frac{AP}{BP} = \frac{LR}{MR},$$

that is, $\quad \dfrac{n_1}{n_2} = \dfrac{x - x_1}{x_2 - x}$; whence $x = \dfrac{n_2x_1 + n_1x_2}{n_1 + n_2}$.

Similarly $\quad y = \dfrac{n_2y_1 + n_1y_2}{n_1 + n_2}$.

The axes may be inclined at any angle.

In the particular case of bisection $n_1 = n_2$, and thus we see that the middle point of AB is

$$\left(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2}\right).$$

13. In Art. 12 the division was supposed *internal;* but if $n_1 > n_2$, AB can be divided *externally* in the ratio $n_1 : n_2$; that is, a point Q can be found in AB produced, such that

$$AQ : BQ = n_1 : n_2.$$

If $n_1 < n_2$, a point Q' can be found BA produced, such that

$$AQ' : BQ' = n_1 : n_2.$$

The co-ordinates of the point of division will be found to be in either case

$$\frac{n_2x_1 - n_1x_2}{n_2 - n_1}, \ \frac{n_2y_1 - n_1y_2}{n_2 - n_1};$$

which expressions may be deduced from those in Art. 12 by

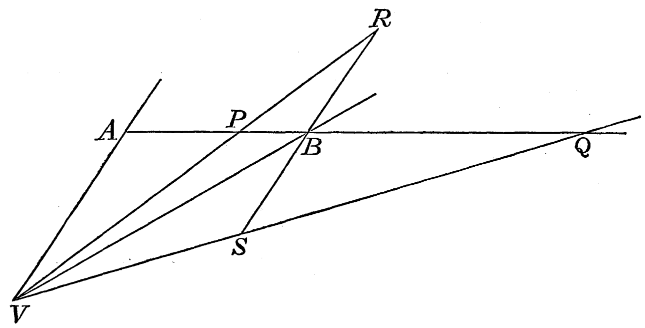
changing the sign of the fraction $\frac{n_1}{n_2}$; for that change of sign changes the x from

$$\frac{x_1 + \frac{n_1}{n_2} \cdot x_2}{\frac{n_1}{n_2} + 1} \text{ to } \frac{x_1 - \frac{n_1}{n_2} \cdot x_2}{-\frac{n_1}{n_2} + 1},$$

that is to $\frac{n_2 x_1 - n_1 x_2}{n_2 - n_1}$; and the y to $\frac{n_2 y_1 - n_1 y_2}{n_2 - n_1}$. Thus we may consider the point Q as a point where AB is divided in the ratio $n_1 : -n_2$, and the point Q' as a point where AB is divided in the ratio $-n_1 : n_2$, placing the negative sign before n_1 or n_2 according as the corresponding segment of the line AB is measured along AB forwards or backwards.

14. The following construction may be used for finding a point Q in AB produced, such that $AQ : BQ = AP : BP$ (AP being of course greater than BP, since AQ is necessarily greater than BQ).

Take any point V not in the line AB; join VA; through B



draw RBS meeting VP produced in R; make BS equal to BR; and join VS meeting AB produced in Q.

For $\frac{BP}{AP} = \frac{BR}{AV}$ (by similar triangles BPR, APV)

$$= \frac{BS}{AV} = \frac{BQ}{AQ} \text{ (by similar triangles } BQS,\ AQV).$$

15. When a line AB is divided in P and Q so that $AP : BP = AQ : BQ$, it is said to be divided *harmonically*. For, by the definition of Harmonical Progression, AP, AB, AQ are in Harmonical Progression, since the first AP is to the third AQ as the difference PB of the first and second to the difference BQ of the second and third.

(We have supposed AP greater than BP. If AP be less than BP, Q lies in BA produced, and BP, BA, BQ are in Harmonical Progression.)

QP is also divided harmonically at B and A, for

$$QB : BP = QA : PA.$$

Thus QB, QP, QA are in Harmonical Progression.

Examples on Chapter I.

N.B. The ω in brackets at the end of a question means that oblique co-ordinates are to be used.

1. Find the polar co-ordinates of the points whose rectangular co-ordinates are as follows:

(1) $x = 1, y = 1$; (2) $x = -1, y = -2$;
(3) $x = -3, y = 0$; (4) $x = 0, y = -4$;

and indicate the points in a figure.

2. Find the rectangular co-ordinates of the points whose polar co-ordinates are as follows:

(1) $r = 2, \theta = \frac{\pi}{6}$; (2) $r = 1, \theta = 0$;

(3) $r = -1, \theta = -\frac{\pi}{6}$; (4) $r = -1, \theta = 0$;

and indicate the points in a figure.

3. Find the distance between the points $(-8, -1)$, $(5, -5)$; and the perimeter of the triangle formed by joining the first three points in Example 1.

4. Find the area of the triangle whose angular points are $(0, 1)$, $(-1, 0)$, $(2, -2)$.

5. The co-ordinates of A, B, C are respectively

$$x_1 y_1, \ x_2 y_2, \ x_3 y_3,$$

and the middle points of BC, CA, AB are D, E, F. Find the co-ordinates of D, E, F, and those of the middle points of the sides of the triangle D, E, F................................(ω).

6. Find also the co-ordinates of the point which divides AD in the ratio $2:1$, and apply the result to shew that the lines AD, BE, CF meet in a point................................(ω).

7. Prove geometrically that the distance between the points (r, θ), (r', θ') is $\sqrt{r^2 + r'^2 - 2rr' \cos(\theta \sim \theta')}$, and that the area of the triangle formed by joining them to each other and to the origin is $\frac{1}{2}rr' \sin(\theta \sim \theta')$.

8. ABC is *any* triangle, and AB, AC are taken for co-ordinate axes. Given that $AB = c$, and $AC = b$, find the co-ordinates of the middle point of BC, and the distance of that point from the origin.

9. Apply Example 8 to prove that in any triangle the squares on two sides are together double of the square on half the base and on the line joining the vertex to the middle point of the base.

10. The axes being inclined at an angle ω, and the origin and axis of x being made the origin and initial line of a system of polar co-ordinates, the formulæ for transforming Cartesian into polar co-ordinates are

$$x = r\frac{\sin(\omega - \theta)}{\sin\omega}, \quad y = r\frac{\sin\theta}{\sin\omega}.$$

N.B. 'Cartesian' co-ordinates (so named after Des Cartes) are such as determine the position of a point by reference to two fixed straight lines.

11. If F, F' be the middle points of AB and PQ in Art. 15, $FP \,.\, FQ = FA^2$, and $F'A \,.\, F'B = F'P^2$.

Trace the changes in Q's position as P moves from B to A.

12. Obtain a general formula for all the names of the point (r, θ) in polar co-ordinates.

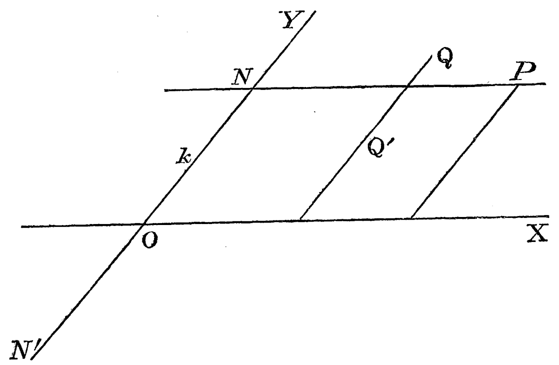
CHAPTER II.

THE STRAIGHT LINE.

N.B. When 'co-ordinates' are spoken of it is generally understood that they are Cartesian co-ordinates (see note on Ex. 10, Chap. I.).

In this Second Chapter the inclination of the axes is in general unrestricted.

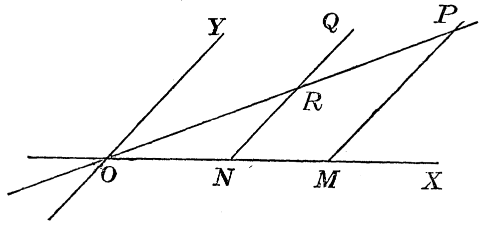
16. The axes being inclined at any angle, let xy be any point P on a straight line NP, which is parallel to OX and meets OY at a point N, distant k from O. Then $y = k$ (Euclid I. 34). This is true for *any* position of P in the line, and is not true for any point out of the line; for the y of a point



Q or Q' out of the line must be either algebraically greater or algebraically less than k. Thus the equation $y = k$ belongs peculiarly to the line NP, and may be called the 'equation of the line NP.' Any one acquainted with the system of co-ordinates could lay

down on the plane of operation the line NP from the datum that it is 'the line $y = k$.' The equation $y = k$ *represents* the line NP. So the equation $y = -k$ represents a line parallel to OX and on the opposite side of OX from NP, meeting the axis of Y in a point N' such that $ON' = ON$. In like manner the lines $x = h$, $x = -h$ are parallel to, and equidistant from, the axis of y. Again, the equation $2y + 3 = 0$ represents a line parallel to the axis of x and meeting the 'negative' part of the axis of y at a distance $\frac{3}{2}$ from the origin. And, generally, the equation $Ay + B = 0$ represents a line parallel to the axis of x, and the equation $Cx + D = 0$ represents a line parallel to the axis of y. That is, a *simple* (or 'linear') equation involving only one of the 'variables' x, y represents a line parallel to the axis of the omitted variable.

17. Again, let Py be any point P on a straight line passing through the origin. Let OM, PM be the co-ordinates of P.



Then, by Euclid, Bk. VI., the ratio $\frac{PM}{OM}$ is the same for every position of P in the line. Let us call it m. Then the following relation exists between the co-ordinates xy of any point in the line

$$\frac{y}{x} = m, \text{ or } y = mx \dots\dots\dots\dots\dots\dots (1);$$

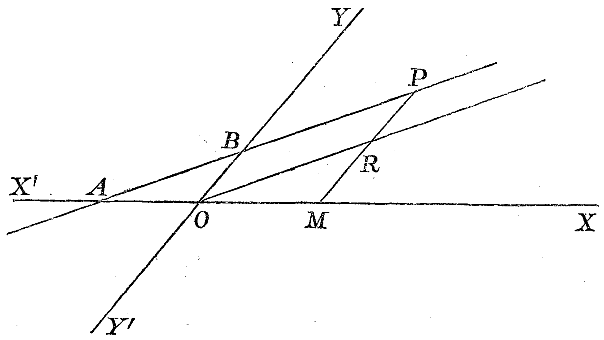
and this relation does not exist between the co-ordinates of any point Q not in the line. For let the ordinate QN meet OP in R, then $\frac{QN}{ON}$ $\left(\text{the } \frac{y}{x} \text{ of } Q\right)$ is not equal to $\frac{RN}{ON}$, that is, not equal to m. Hence the equation (1) is peculiarly character-

istic of the line OP, is, in fact, enough to identify the line OP, and is called the 'equation to the line OP.'

18. In these cases the quantities h, k, m are called *constants*, just as x, y are called *variables*. They are constant so long as we keep to the same line, but different for different lines. The constant m in the equation $y = mx$ is numerically the ratio of the sines of the angles which the line makes with the axes. The *sign* of m determines the compartment in which OP lies. If m be positive, the line lies in the first and third compartments; if negative, in the second and fourth.

When the axes are rectangular m is the tangent of the angle which that part of the line which is on the OY side of the axis of x makes with the axis of x produced in a positive direction. Thus, to speak accurately, the lines $y = x$, $y = -x$ make angles 45° and 135° with the axis of x.

19. Again, let us take a line AB which meets the axes in A and B, not passing through O nor parallel to either axis. Let P be



any point xy in this line. Draw PM parallel to OY, meeting OX in M, and draw OR parallel to AB, meeting PM in R. Suppose $OB = c$, and let m be, as before, the ratio of the sines of the inclinations of AP to the axes of x and y, so that $y = mx$ is the equation to OR.

Then PM, the y of P, exceeds RM, the y of R, by PR, or OB, or c. That is, at the point P

$$y = mx + c.$$

This equation is true at every point of the line AB, and (as the reader can prove) at no point out of the line. It is therefore peculiarly the property of the line AB, and is enough to identify the line AB. It is called the equation to the line AB.

Similarly the line $y = mx - c$ is parallel to the line $y = mx$, and is equidistant with AB from that line.

20. We have now shewn that corresponding to every straight line there is an equation which can be put in one of the following forms:

$$x = h \;(1),\quad y = k \;(2),\quad y = mx \;(3),\quad y = mx + c \;(4).$$

That is, by giving proper values to the constant or constants, we can find among (1), (2), (3) and (4) the equation to the line. Thus for the axis of x choose form (2) and put $k = 0$. For the axis of y choose form (1) and put $h = 0$. It is to be observed that the form (4) includes (2) and (3).

All the forms are included in the general equation of the first degree,

$$Ax + By + C = 0.$$

In other words, this last equation can be made to represent any proposed straight line by giving proper values to A, B and C.

21. The question now arises, whether *curved* lines have equations corresponding to them and representing them. It is more convenient to ask whether to equations of higher degrees than the first there correspond curved, instead of straight, lines. The answer is 'yes, generally:' but we must for the present confine ourselves to the straight line, laying down, however, certain definitions applicable throughout the subject of Plane Co-ordinate Geometry.

An equation is said to represent, or be the equation to a line, straight or curved, when the equation is satisfied by the co-ordinates of every point in the line and by the co-ordinates of no other point.

And the line, straight or curved, is called the *locus* of the equation.

An equation is said to be of the n^{th} degree when, after it has been reduced to a rational and integral form as far as the variables are concerned, the term or terms of highest dimensions in the variables are of n dimensions.

Thus the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, reduced to an integral form, is

$$x+y=\frac{xy}{a},$$

in which $\frac{xy}{a}$, being of two dimensions in x and y, and being the term of highest dimensions, shews the equation to be of the second degree.

So $x^{-2}+\sqrt{y}=\sqrt{a}$, reduced to a rational and integral form, is $yx^4=(x^2\sqrt{a}-1)^2$, and is therefore of the fifth degree.

An equation of the first degree, reduced to a rational and integral form as far as the variables are concerned, is called a *linear* equation. Thus $\frac{x}{3}+y\sqrt{2}-1=0$ is a linear equation. The general type of such equations is $Ax+By+C=0$, and $Ax+By+C$ is the general type of *linear functions* of x and y.

22. We may prove directly that an equation of the first degree can represent no line but a *straight* line. For let x_1y_1, x_2y_2, x_3y_3 be any three points on the *locus* of the equation

$$Ax+By+C=0,$$

then

$$Ax_1+By_1+C=0,\quad Ax_2+By_2+C=0,\quad Ax_3+By_3+C=0.$$

From these three equations eliminate A, B, C by cross multiplication, that is, multiply them respectively by y_2-y_3, y_3-y_1, y_1-y_2 and add. There results

$$A\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}=0,$$

or, since A is not supposed zero,

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

This equation informs us that the area of the triangle of

which the vertices are x_1y_1, x_2y_2, x_3y_3 is zero. Now these three points are *any* three points on the locus, and, as they are proved to lie in a straight line, the locus is a straight line.

If A be zero, it is evident that the y of every point in the locus is the same: hence the locus must be a straight line parallel to the axis of x. If B be zero, the locus is a straight line parallel to the axis of y.

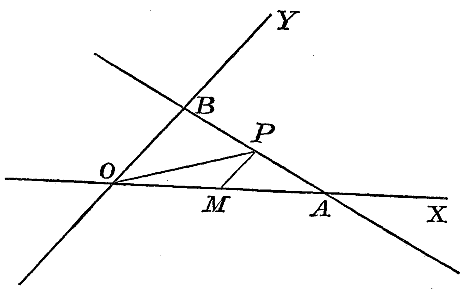
Thus the equation $Ax + By + C = 0$, whatever be the values of A, B, C, represents a straight line, and if the $Ax + By + C$ of a point vanishes, that point lies on the line, and if the $Ax + By + C$ of a point does not vanish, that point does not lie on the line. Thus the $3x - y + 1$ of the point (1, 4) vanishes, and the $3x - y + 1$ of the point (2, 3) does not vanish. Therefore the point (1, 4) lies on the line $3x - y + 1 = 0$, and the point (2, 3) does not lie on that line.

We shall sometimes speak of the line represented by the equation $Ax + By + C = 0$ as 'the line $Ax + By + C$.'

23. To find the equation to a straight line in terms of the intercepts which it makes on the co-ordinate axes.

Let P be any point xy on a line AB which meets the axes OX, OY in A, B. Let $OA = a$, $OB = b$. Join OP and draw the ordinate PM. Then

$$\Delta AOB = \Delta BOP + \Delta AOP.$$



But $$\frac{\Delta BOP}{\Delta AOB} = \frac{BP}{AB} \text{ (Euclid, VI. 1)} = \frac{OM}{OA} \text{ (VI. 2)} = \frac{x}{a},$$

and similarly $$\frac{\Delta AOP}{\Delta AOB} = \frac{y}{b}.$$

Therefore $$\frac{x}{a} + \frac{y}{b} = 1.$$

This is the equation required.

24. If we suppose a infinite, the equation becomes $y = b$, representing a line through B parallel to Ox. Similarly the equation $\frac{x}{a} + \frac{y}{b} = 1$ includes the form $x = a$. When a and b are both very great the line is at a very great distance. Also the portion AB, in any case, lies in the 1st, 2nd, 3rd, or 4th compartments according as the signs of a and b are $+\,+$, $-\,+$, $-\,-$, or $+\,-$.

Ex. Draw the straight line $2x - 3y + 7 = 0$. The reduced form is

$$\frac{x}{\frac{-7}{2}} + \frac{y}{\frac{7}{3}} = 1.$$

Thus the portion AB lies in the 2nd compartment.

25. The intercepts of any line $A'B'$ parallel to AB are proportional to a and b. Let them be μa, μb. Then the equation to $A'B'$ is

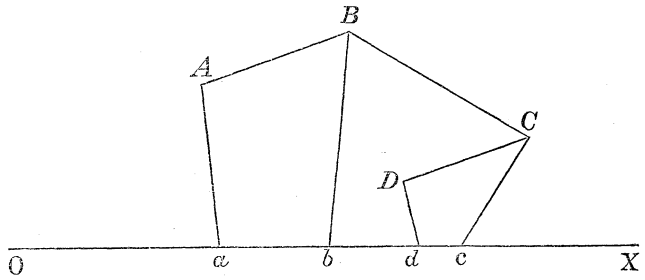
$$\frac{x}{\mu a} + \frac{y}{\mu b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{or} \quad \frac{x}{a} + \frac{y}{b} = \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let AB be fixed and suppose μ to change from $+\infty$ to $-\infty$. $A'B'$ is at first infinitely distant, and then approaches nearer and nearer to AB, coincides with AB when $\mu = 1$, passes through the origin as appears from (2) when $\mu = 0$, and finally is lost at infinity on the origin side of AB.

Thus a line passing through the origin has infinitely small intercepts, but they may have a finite ratio.

26. Let Ox be a straight line chosen in space, and let A, B, C, D be any points in space. Draw (Euclid, I. 12) Aa,



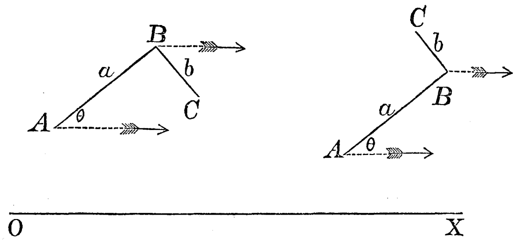
Bb, Cc, Dd perpendicular to OX. Then the points a, b, c, d are called the *projections* of the points A, B, C, D on OX, and the lines ab, bc, ad, &c. are called the *projections* of the lines AB, BC, AD, &c. on OX. Let the direction OX be considered positive and the direction XO negative: so that ab, bc, ac, dc are, in our figure, measured in the positive direction, and ba, cb, ca, cd in the negative direction.

Then ad is the algebraic sum of ab, bc, cd: or, the projection of the line joining A to D is the algebraic sum of the projections of the three lines which join A to B, B to C, C to D.

And generally the projection of the line joining a point A to a point B is the algebraic sum of the projections of the lines AA_1, A_1A_2, $A_2A_3 \ldots A_{n-1}A_n$, A_nB. The order of the two letters which denote any line in this series must be strictly observed. The projection of AD on OX is not the sum of the projections of AF, DF but the sum of the projections of AF, FD. So also the projection of DA on OX is the sum of the projections of DF, FA.

The sum of the projections of the sides of any closed polygon is zero: for if we set out from A and travel round to A again through any number of points, the projection a also returns to its original position, and has accomplished, therefore, a distance zero.

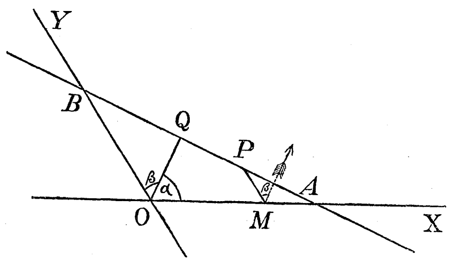
Now suppose the line OX and the points projected to lie all in one plane. Then ab the projection of $AB = AB \times$ cosine of the angle between AB and OX. If this angle be acute, then ba the projection of $ba = BA \times$ cosine of the angle between BA and OX, and (as this angle is obtuse, being the supplement of the former angle) ba is negative, as it ought to be.



Let AB, BC, two straight lines of lengths a, b, and at right angles, connect the points A, C. Let the angle between AB and OX be θ. Then the projection of AC on $OX = a \cos\theta + b \sin\theta$ or $a \cos\theta - b \sin\theta$, according as the right angle ABC opens, or does not open, upon OX. (The dotted lines and arrows in the figure indicate the direction of OX.)

Projections in one plane can also be made obliquely, by ordinates drawn in *any* constant direction; but the most usual system of projection is the *orthogonal*, as that which we have touched on is named.

27. The equation to a straight line can be expressed in



terms of the line's distance from the origin, and the angles which this distance makes with the axes.

Let xy be any point P (OM, PM) on a line AB, and let OQ, the perpendicular from O on AB, be of length p, and make with the axes OX, OY the angles α, β. The axes are supposed to contain any angle ω, so that $\alpha + \beta = \omega$. Then OQ is the projection on OQ of the 'broken line' OMP, and, as the angle between MP and OQ is β, this projection

$$= x \cos\alpha + y \cos\beta.$$

Thus

$$x \cos\alpha + y \cos\beta = p \text{ .... } (1)$$

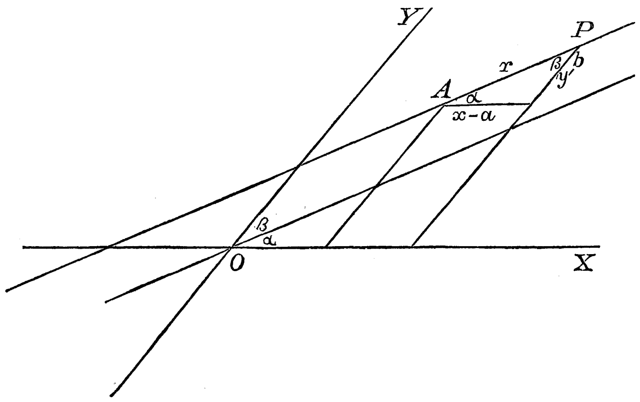
for any point xy in the line; and therefore this equation is the equation to the line.

In the case of rectangular axes $\omega = \frac{\pi}{2}$, and (1) becomes

$$x \cos\alpha + y \sin\alpha = p \text{ .... } (2).$$

In the figure the line crosses the first compartment. If p be negative, the line crosses the 3rd compartment; if $\cos\alpha$ be negative and p and $\sin\alpha$ positive, the line crosses the 2nd compartment (Art. 24).

28. Let xy be *any* point P in a straight line, and let ab be a



fixed point A in the line. Let $AP = r$, and let r be capable of

sign, so that, the line being given and A given in the line, any point in the line can be found if its r be known. Let the line make angles α, β with the axes, so that, if ω be the angle between the axes, $\alpha + \beta = \omega$. If the parallel line through O lie in the angle $X'OY$, β may be considered negative.

Then $$\frac{x-a}{r} = \frac{\sin\beta}{\sin\omega}, \quad \frac{y-b}{r} = \frac{\sin\alpha}{\sin\omega},$$

by Trigonometry.

Let $\frac{\sin\beta}{\sin\omega}$, $\frac{\sin\alpha}{\sin\omega}$ be denoted by l, m; so that l, m are ratios of the projections of any portion of the line on the co-ordinate axes to the portion projected (the projections being made by lines parallel to the axes).

Then $$\frac{x-a}{l} = \frac{y-b}{m} = r \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is another form of the equation to a straight line. We have supposed r positive: if in the figure P had been taken on the other side of A, then $x-a$, $y-b$, and r would have been negative. The result (1) would have been unaffected.

29. The angles α, β may be called *direction-angles* of the line AP or of any parallel line, and l, m the *direction-ratios* of the line AP, or of any parallel line. The direction of AP may be called the direction $[l, m]$. Thus AP is drawn through ab in direction $[l, m]$.

Let L, M be any quantities proportional to l, m. Then AP may be represented by the equation

$$\frac{x-a}{L} = \frac{y-b}{M},$$

but these equal quantities are not equal to r, unless L, M are *equal* to l, m as well as proportional. Thus we may also call the direction $[l, m]$ 'the direction (L, M)', the square brackets being reserved for the special case of direction-*ratios*.

When the axes are rectangular, $\alpha+\beta=\frac{\pi}{2}$, and l, m are the cosines of the direction-angles, or the *direction-cosines.* The line $y=2x$ is the line (1, 2) or (2, 4) or (f, $2f$), but (as the direction-angles are $\tan^{-1} 2$ and the complement of $\tan^{-1} 2$) $\left[\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$. So the direction ($A$, B) is the direction $\left[\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right]$.

Again, let the axes be inclined at any angle, and let O and Ox be pole and initial line. Then if $[l, m]$ be the direction of OP, the equation to OP is $\frac{x}{l}=\frac{y}{m}=r$, a convenient formula for changing Cartesian into polar co-ordinates.

30. *To find the direction-angles of the line* $y=mx+c$, *the axes being inclined at an angle* ω.

Let AB (fig. to Art. 19) be the line $y=mx+c$, and OR the parallel line $y=mx$, and let RM, parallel to OY, meet OX in M. Also let $\angle ROX=\alpha$, and $\angle ROY=\beta$. Then α, β are the direction-angles required, and $\alpha+\beta=\omega$.

From the triangle ROM

$$\frac{\sin\alpha}{\sin\beta}=\frac{\sin ROM}{\sin ORM}=\frac{RM}{OM}=m.$$

Thus $\sin\alpha=m\sin(\omega-\alpha)=m(\sin\omega\cos\alpha-\cos\omega\sin\alpha)$,

$$\text{or}\quad \sin\alpha(1+m\cos\omega)=\cos\alpha\,.\,m\sin\omega,$$

$$\text{or}\quad \tan\alpha=\frac{m\sin\omega}{1+m\cos\omega}.$$

Therefore $\left(\text{since if } \tan\theta=\frac{a}{b},\ \ \sin\theta=\frac{a}{\sqrt{a^2+b^2}},\ \text{ and } \cos\theta=\frac{b}{\sqrt{a^2+b^2}}\right)$

$$\sin\alpha=\frac{m\sin\omega}{\sqrt{1+2m\cos\omega+m^2}},\quad \cos\alpha=\frac{1+m\cos\omega}{\sqrt{1+2m\cos\omega+m^2}},$$

and

$$\sin\beta = \frac{\sin\omega}{\sqrt{1+2m\cos\omega+m^2}},\quad \cos\beta = \frac{m+\cos\omega}{\sqrt{1+2m\cos\omega+m^2}}.$$

Polar Co-ordinates.

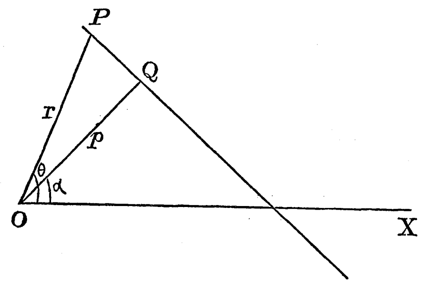
31. Let $Ax+By+C=0$ be the equation to any straight line in rectangular co-ordinates, and let r, θ be the polar co-ordinates of any point xy in the line.

Then $x = r\cos\theta$, $y = r\sin\theta$, and $Ax+By+C=0$.

Therefore $$Ar\cos\theta + Br\sin\theta + C = 0 \text{ .... (1).}$$

This is a relation connecting the r and θ of any point in the line, and is therefore the polar equation to the line.

We may also obtain geometrically a form of the equation.



Let P be any point (r, θ) in the line and let OQ be the perpendicular on the line from the origin. Let $OQ = p$, and let $\angle\, QOX = \alpha$. Then from the triangle OPQ we see that

$$p = r\cos(\theta-\alpha) \text{ ....(2).}$$

This is also an equation to the line, and is of the form (1), being

$$\cos\alpha \,.\, r\cos\theta + \sin\alpha \,.\, r\sin\theta - p = 0.$$

The polar equation to a straight line drawn through the pole is of the form $\theta = \text{constant}$. This is geometrically manifest.

Examples on Chapter II.

1. Of what degree in x and y is the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\,?$$

2. Find the intercepts made on the axes by the lines

$$2x - 3y = 5, \quad Ax + By + C = 0 \ldots\ldots\ldots\ldots\ldots(\omega).$$

3. Determine the inclinations to Ox of the lines

$$x + y = 0, \quad px + qy = r, \quad \sqrt{3}y + x + 2 = 0.$$

4. Find the direction-cosines of the lines

$$x + y = 5, \quad 2x + y = 3, \quad x + 2y + 6 = 0,$$

$$x + \sqrt{3}y = 2, \quad \sqrt{3}x - y + 5 = 0,$$

$$fx + gy = 0, \quad fx - gy = 0, \quad Ax + By + C = 0.$$

5. If $[l, m]$ be the direction of a line, and ω the inclination of the axes,

$$l^2 + m^2 + 2lm \cos \omega = 1.$$

6. If α, β be the direction-angles of any line,

$$\cos^2 \alpha + \cos^2 \beta - 2 \sin \alpha \sin \beta \cos \omega = \sin^2 \omega.$$

7. Determine the inclinations of the lines

$$y = x, \quad x + y \cos \omega = 0, \quad Ax + By + C = 0,$$

to the axis of x .... (ω).

8. Find the direction-ratios of the line $y = 2x + 3$, the axes being inclined at $45°$.

Also find those of $Ax + By + C = 0$, for any inclination of the axes.

9. Transform $x - y = 3$ to polar co-ordinates and

$$r \cos\left(\theta + \frac{\pi}{6}\right) + 5 = 0$$

to rectangular.

10. Draw the straight line

$$r\cos\left(\theta+\frac{\pi}{4}\right)+\sqrt{2}=0,$$

and determine whether the point $x=2$, $y=4$ lies upon it.

11. Draw all the straight lines numerically given in Example 4.

12. Prove that if $\frac{A}{A'}=\frac{B}{B'}$, and in no other case, the lines $Ax+By+C$, $A'x+B'y+C'$ are parallel (see Art. 23)......(ω).

CHAPTER III.

THE STRAIGHT LINE.

32. (1) THE following Lemma from Algebra is easily proved and should be remembered:

If $ax + by + cz = 0$, and $a'x + b'y + c'z = 0$, then

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

(2) Another algebraic theorem which is useful in Geometry is this:

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = $ &c., then each fraction

$$= \frac{\lambda a_1 + \mu a_2 + \nu a_3 + \dots}{\lambda b_1 + \mu b_2 + \nu b_3 + \dots} = \sqrt[n]{\frac{\lambda a_1^n + \mu a_2^n + \dots}{\lambda b_1^n + \mu b_2^n + \dots}}$$

in short, $= \frac{\phi(a_1, a_2, a_3 \dots a_n)}{\phi(b_1, b_2, b_3 \dots b_n)}$, ϕ being *any* homogeneous function of the first degree.

Any given case of this theorem can be proved by supposing

$$k = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \text{\&c.}$$

so that b_1k, $b_2k \dots$ may be substituted for a_1, $a_2 \dots$.

33. *To find the co-ordinates of the point of intersection of two given lines*

$$Ax + By + C = 0 \dots (1), \quad A'x + B'y + C' = 0 \dots (2).$$

Let P be the point of intersection. Then the $Ax+By+C$ of $P=0$, because P is on the line (1), and the $A'x+B'y+C'$ of $P=0$, because P is on the line (2).

In other words, at the point P

$$Ax+By+C\,.\,1=0, \text{ and } A'x+B'y+C'\,.\,1=0.$$

Therefore by Lemma (1),

$$\frac{x}{BC'-B'C}=\frac{y}{CA'-C'A}=\frac{1}{AB'-A'B}.$$

Thus the lines $Ax+By+C$, $A'x+B'y+C$ meet in the point

$$\left(\frac{BC'-B'C}{AB'-A'B}, \frac{CA'-C'A}{AB'-A'B}\right), \text{ and}$$

Given the equations to two straight lines, the co-ordinates of their point of intersection are found by combining the given equations.

Ex. The lines $x=h$, $y=k$ meet in the point (h, k). Thus when we speak of the point $x=h, y=k$, we give two lines which by their intersection determine the point.

34. Euclid's definition of parallel straight lines may be altered to this: Parallel straight lines are such as are in the same plane and do not meet unless infinitely produced.

Thus the lines $Ax+By+C$, $A'x+B'y+C'$ are parallel if their point of intersection be infinitely distant; that is, if

$$AB'-A'B=0, \text{ or } \frac{A}{A'}=\frac{B}{B'}.$$

Ex. The lines $2x+3y=0$ and $2x+3y=5$ are parallel, for here

$$\frac{A}{A'}=1=\frac{B}{B'}.$$

So the lines $2x+3y=1$, $4x+6y+5=0$ are parallel, and generally, two lines are parallel when the x and y terms in their equations are *virtually* the same. Thus $Ax+By+C=0$ and $\lambda(Ax+By)+C'=0$ are parallel.

Thus the line $Ax + C = 0$ is parallel to the line $x = 0$, that is, to the axis of y.

Again, the lines $y = mx + c$, $y = mx + c'$ are parallel, as may also be proved from the equality of their direction-angles.

35. *To find the condition that the lines* $\mathrm{Ax + By + C} \ldots (1)$, $\mathrm{A'x + B'y + C'} \ldots (2)$, $\mathrm{A''x + B''y + C''} \ldots (3)$, *may meet in a point.*

Since (1) is to pass through the intersection of (2) and (3), the co-ordinates of the intersection of (2) and (3) may satisfy (1). That is (Art. 33),

$$A\,(B'C'' - B''C') + B\,(C'A'' - A''A') + C\,(A'B'' - A''B') = 0.$$

This is the required condition, and is the result of eliminating x and y from the equations to the three lines.

Ex. The lines $y = mx$, $y = 0$, $x = 0$ meet in a point. So do the lines

$$Ax + By + C,\ A'x + B'y + C,\ Ax + By + C - \lambda\,(A'x + B'y + C'),$$

but this example deserves a separate examination.

36. Let $Ax + By + C = 0 \ldots (1)$, $A'x + B'y + C' = 0 \ldots (2)$ be equations to two given lines. Then

$$Ax + By + C = \lambda\,(A'x + B'y + C') \ \ldots\ldots\ldots\ldots (3)$$

is of the first degree and therefore represents *some* straight line.

Also the point of intersection of (1) and (2) lies on this line, for since the $Ax + By + C$ and the $A'x + B'y + C'$ of that point are both zero, the co-ordinates of that point, substituted for x and y in (3), give $0 = 0$.

Thus (3) represents *some* straight line passing through the point of intersection of (1) and (2). Call that point P: then by giving different values to λ, we get different straight lines through P.

But can we by giving a value to λ make (3) represent *any*

straight line through P? Yes: for let PQ be any such line and let it be required to determine λ so that (3) shall represent the line PQ. Let x_1, y_1 be the co-ordinates of Q.

Since Q is to lie on (3) we must have

$$Ax_1 + By_1 + C = \lambda (A'x_1 + B'y_1 + C').$$

Thus the value to be given to λ is $\dfrac{Ax_1 + By_1 + C}{A'x_1 + B'y_1 + C'}$, and the equation to PQ is $\dfrac{Ax + By + C}{A'x + B'y + C'} = \dfrac{Ax_1 + By_1 + C}{A'x_1 + B'y_1 + C'}$, which is *visibly* satisfied by the values x_1, y_1 of x, y.

Ex. $y = mx$ is the equation to a line through the intersection of $y = 0$, $x = 0$, and by giving a proper value to m can be made to represent *any* straight line through the origin.

The equation to the line joining x_1, y_1 to the origin is $\dfrac{y}{x} = \dfrac{y_1}{x_1}$ or $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

37. It has been proved that *the equation*

$$Ax + By + C = \lambda (A'x + B'y + C)$$

can by giving the right value to λ *be made the equation to any straight line passing through the intersection of* $Ax + By + C$ *and* $A'x + B'y + C$. Conversely, *the equation to any line through that point of intersection must be of the form*

$$Ax + By + C = \lambda (A'x + B'y + C').$$

38. *To find the equation to the line through two given points* x_1, y_1 *and* x_2, y_2.

The first point is the intersection of the lines $x = x_1$, $y = y_1$; thus the required equation must be of the form

$$\frac{x - x_1}{y - y_1} = \text{a constant} \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}.$$

Now x_2, y_2 lies on the line; therefore x_2, y_2 satisfy (1); that is,

$$\frac{x_2 - x_1}{y_2 - y_1} = \text{the constant}.$$

Thus the required equation is

$$\frac{x-x_1}{y-y_1}=\frac{x_2-x_1}{y_2-y_1} \text{ or } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \text{.........(2).}$$

Another form is of course $\frac{x-x_2}{x_1-x_2}=\frac{y-y_2}{y_1-y_2}$ ....(3).

Another is

$$x(y_1-y_2)+y(x_2-x_1)+x_1y_2-x_2y_1=0 \text{.........(4),}$$

which asserts that the area of the triangle whose vertices are x_1y_1, x_2y_2 and xy is zero.

Up to this point of Chapter III. the axes may have been inclined at any angle. We shall now suppose them rectangular until further notice.

39. The equation $Ax+By+C=0$ (1) can be reduced to the form $x\cos\alpha+y\sin\alpha-p=0$ (2).

For (1) and (2) are identical if $\frac{\cos\alpha}{A}=\frac{\sin\alpha}{B}=\frac{-p}{C}$, in which case by Lemma (2) each ratio $=\sqrt{\frac{\cos^2\alpha+\sin^2\alpha}{A^2+B^2}}=\frac{1}{\sqrt{A^2+B^2}}$.

Thus $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha=\frac{B}{\sqrt{A^2+B^2}}$, and $p=\frac{-C}{\sqrt{A^2+B^2}}$.

The rule for reducing any equation of the form (1) to the form (2) is therefore this: Divide by the square root of the sum of the squares of the coefficients of x and y.

Thus $2x-y+5=0$ becomes $\frac{2x}{\sqrt{5}}-\frac{1}{\sqrt{5}}+\sqrt{5}=0$.

Also the length of the perpendicular from the origin on the line is the numerical value of $\frac{C}{\sqrt{A^2+B^2}}$, and the direction-cosines of this perpendicular are $\frac{A}{\sqrt{A^2+B^2}}$, $\frac{B}{\sqrt{A^2+B^2}}$.

40. *To find the angle between two lines whose directions are* [l, m], [l′, m′].

If α, α' be the angles whose cosines are l, l', $\alpha \sim \alpha'$ is the angle required, and $\cos(\alpha \sim \alpha') = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha' = ll' + mm'$.

The angle between the directions (l, m), (l', m') is the angle between $\left[\frac{l}{\sqrt{l^2+m^2}}, \frac{m}{\sqrt{l^2+m^2}}\right]$ and $\left[\frac{l'}{\sqrt{l'^2+m'^2}}, \frac{m'}{\sqrt{l'^2+m'^2}}\right]$ and its cosine is
$$\frac{ll'+mm'}{\sqrt{(l^2+m^2)(l'^2+m'^2)}}.$$

41. *To find the angle between the lines* $Ax + By + C$, $A'x + B'y + C'$.

The parallels through the origin, which are inclined at the same angle as the given lines, are $Ax + By$, $A'x + B'y$ or
$$\frac{x}{B} = \frac{y}{-A}, \quad \frac{x}{B'} = \frac{y}{-A'},$$
and these are in directions
$$\left[\frac{B}{\sqrt{A^2+B^2}}, \frac{-A}{\sqrt{A^2+B^2}}\right], \quad \left[\frac{B'}{\sqrt{A'^2+B'^2}}, \frac{-A'}{\sqrt{A'^2+B'^2}}\right].$$

The cosine of the angle between them is therefore by (9)
$$\frac{AA'+BB'}{\sqrt{(A^2+B^2)(A'^2+B'^2)}}.$$

42. *To find the condition that the lines* $Ax + By + C$, $A'x + B'y + C'$ *may be at right angles.*

The cosine of the angle between the lines is the cosine of 90^0, and therefore vanishes. Thus the condition is
$$AA' + BB' = 0.$$

The directions (l, m), (l', m') are at right angles if
$$ll' + mm' = 0. \text{ (Art. 40.)}$$

43. Two lines which are not at right angles make an acute and an obtuse angle, of which the cosines are numerically equal, but in sign opposite. Thus if we wish for the cosine of the *obtuse* angle in Art. 41, we must give to the radical the opposite sign to that of $AA' + BB'$.

44. From Art. 41, or from a figure (see Euclid, I. 32), we can prove that the angle between $y = mx + c$, and $y = m'x + c'$ is

$$\tan^{-1}\frac{m \sim m'}{1 + mm'}.$$

The condition of perpendicularity is $1 + mm' = 0$, for in this case the tangent of the angle is infinite.

45. If $A'x + B'y + C'$ be perpendicular to $Ax + By + C$, then $\frac{A'}{B} = \frac{B'}{-A}$. Thus $A'x + B'y + C' = 0$ can be put in the form $Bx - Ay =$ a constant, and this is the general form of equation to a straight line perpendicular to a given straight line $Ax + By + C$. The x and y terms are deduced from those of the given equation by this rule: *Interchange or invert the coefficients of* x *and* y, *and alter the sign of one of them.*

Ex. To find the equation to a line through the point $x'\,y'$ perpendicular to the line $\frac{x}{a} + \frac{y}{b} = 1$.

The equation must be of the form $ax - by =$ some constant, and *what* constant is determined by the point $x'\,y'$. For as $x'\,y'$ lies on the line we are drawing, $ax' - by' =$, or *is*, that constant; thus the equation required is $ax - by = ax' - by'$.

In like manner the equation to a line through $x'\,y'$ parallel to $Ax + By + C$ is $Ax + By = Ax' + By'$, and this result holds for any inclination of the axes.

46. *To find the length of the perpendicular from the point* $x'\,y'$ *on the line*

$$\mathrm{Ax} + \mathrm{By} + \mathrm{C} = 0 \quad \text{..........................} (1).$$

The equation to a line through $x'\,y'$ parallel to the given line is $$Ax + By - (Ax' + By') = 0 \ldots\ldots\ldots\ldots(2),$$ and the perpendicular required is equal to the distance between the lines (1) and (2).

Now the distances of (1) and (2) from the origin are, without regard to sign, equal to

$$\frac{C}{\sqrt{A^2+B^2}} \text{ and } \frac{-(Ax'+By')}{\sqrt{A^2+B^2}} \ldots\ldots\ldots\ldots(3).$$

If the lines (1) and (2) lie on the same side of the origin, then C and $-(Ax'+By')$ have the same sign (Art. 24). Also in this case we require the *difference* of the distances from the origin.

If the lines (1) and (2) lie on opposite sides of the origin, then C and $-(Ax'+By')$ have different signs. Also in this case we require the *sum* of the distances (3).

Thus the distance required is in either case (without regard to sign) $\dfrac{Ax'+By'+C}{\sqrt{A^2+B^2}}$, or, to speak more symbolically, the distance is $\pm\dfrac{Ax'+By'+C}{\sqrt{A^2+B^2}}$.

Ex. The distance of $(0, 1)$ from $x - y = 1$ is the numerical value of $\dfrac{-1-1}{\sqrt{2}}$, or is $\sqrt{2}$.

47. *To find the angle between the lines*
$$y = mx + c, \qquad y = m'x + c',$$
the axes being inclined at an angle ω.

By Art. 30, if α, α' be the inclinations of the lines to Ox,

$$\tan\alpha = \frac{m\sin\omega}{1+m\cos\omega}, \quad \tan\alpha' = \frac{m'\sin\omega}{1+m'\cos\omega},$$

and, by Euclid, I. 32, the angle between the lines is $\alpha \sim \alpha'$.

But $\tan(\alpha \sim \alpha') = \dfrac{\tan\alpha \sim \tan\alpha'}{1 + \tan\alpha\tan\alpha'}$.

Thus the tangent of the angle between the lines is

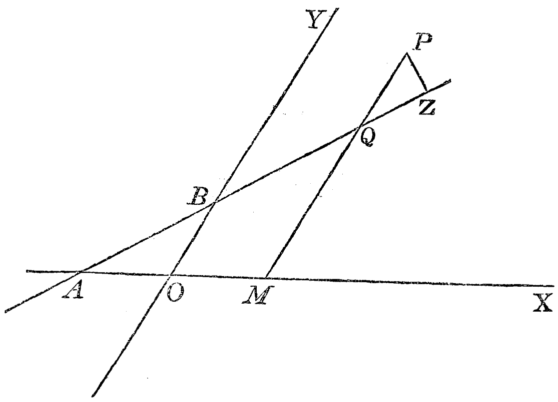
$$\frac{(m \sim m')\sin\omega}{1 + (m + m')\cos\omega + mm'},$$

and the condition of perpendicularity is

$$1 + (m + m')\cos\omega + mm' = 0,$$

or
$$m' = -\frac{1 + m\cos\omega}{\cos\omega + m'}.$$

48. *The axes being inclined at an angle* ω, *to find the distance of the point* x′y′ *from the line* y = mx + c.



Let P be the point and PZ the perpendicular on the given line AB. Draw PQM parallel to OY meeting AB, OX in Q, M.

Then $OM = x'$, and $QM = mx' + c$, and $PM = y'$, therefore

$$PQ = y' - mx' - c.$$

And $PZ = PQ \sin PQZ = (y' - mx' - c)\sin QBY$.

Now by Art. 30,

$$\sin QBY = \frac{\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}},$$

therefore $$PZ = \frac{(y - mx - c)\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}}.$$

Of course the same method could have been used with rectangular axes.

49. *To find the polar equation to the line joining* $(r_1\,\theta_1)$ *and* $(r_2\,\theta_2)$.

Let $$Ar\cos\theta + Br\sin\theta + C = 0 \ldots\ldots\ldots\ldots(1)$$

be the equation required, the ratios of A, B, C being as yet undetermined. Since the points $(r_1\,\theta_1)$, $(r_2\,\theta_2)$ lie on the line,

$$Ar_1\cos\theta_1 + Br_2\sin\theta + C = 0 \ldots\ldots\ldots\ldots(2),$$

$$Ar_2\cos\theta + Br_2\sin\theta + C = 0 \ldots\ldots\ldots\ldots(3).$$

By Lemma (1) we can find from (2) and (3) the ratios of A, B, C. Substitute these in (1). The result of this eliminating process is

$$rr_1\sin(\theta - \theta_1) + r_1 r_2\sin(\theta_1 - \theta_2) + r_2 r\sin(\theta_2 - \theta) = 0 \ldots\ldots(4).$$

Let P be any point (r, θ) on the line, and P_1, P_2 the given points.

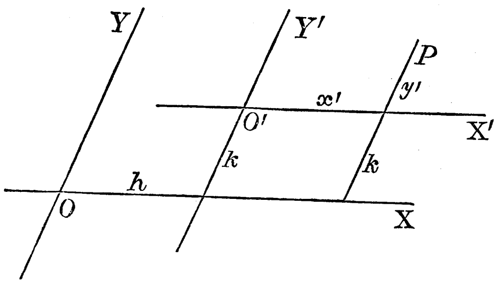
Then equation (4) asserts that of the triangles OPP_1, OP_1P_2, OP_2P, one is equal to the sum of the other two. And from this fact the equation could have been derived.

Transformation of Co-ordinates.

50. It is often convenient to change the origin from the position first fixed upon to some other point in the plane of operations, without altering the direction of the axes.

If h, k be the co-ordinates of the new origin O', referred to the old axes OX, OY, and xy be any point referred to OX, OY,

$x'\,y'$ the same point referred to the new axes $O'X'$, $O'Y'$, then,



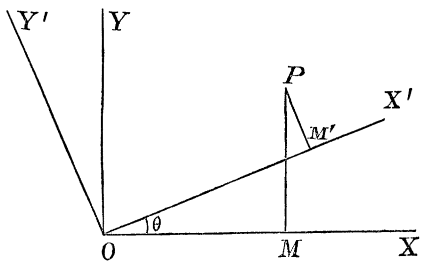
as the reader will easily prove by a figure,

$$x = x' + h,\ \ y = y' + k.$$

Ex. What will be the equation to the line $2x - 3y = 3$ when the origin is transferred to $(1, -2)$? Here $x = x' + 1$, $y = y' - 2$. Therefore $2x - 3y = 2x' - 8y'$ and the new equation is $2x' - 8y' = 3$, or, accents suppressed, $2x - 8y = 3$.

To transform any equation by moving the origin to (h, k), write $x + h$ for x and $y + k$ for y.

51. *To change the axes from one rectangular system to another having the same origin.*



Let x, y be co-ordinates of any point P referred to the old axes OX, OY; x', y' the co-ordinates of P referred to the new axes OX', OY'; and let the angle XOX' be θ. Then OM is the projection of the broken line $OM'P$ on OX, and therefore

$$= OM' \cos\theta - PM' \sin\theta,$$

that is, $$x = x' \cos\theta - y' \sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, by projecting $OM'P$ on OY, we get

$$y = x' \sin\theta + y' \cos\theta \text{.....................(2).}$$

The angle θ was measured towards OY. Thus if OX' had bisected the angle between OY produced backwards and OX, θ would be $-\frac{\pi}{4}$.

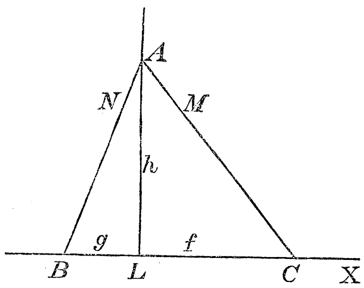
52. Other cases of transformation in Cartesian co-ordinates might be proposed, and some will be found in the examples.

We can, as hinted in Art. 29, transform an equation from Cartesian to Polar co-ordinates by writing lr for x and mr for y, l and m being the direction-ratios of r.

Thus the equation $xy = a^2$ becomes $\frac{a^2}{r^2} = lm$.

53. We shall conclude this chapter with the solution of four problems.

(1) *The lines drawn from the angles of a triangle perpendicular to the opposite sides meet in a point.*



Let ABC be the triangle, and let AL, BM, CN be the three perpendiculars. Take L for origin, and LC, LA for axes of x and y.

Let $LC = f$, $LB = g$, $LA = h$.

Then the equation to AC is $\frac{x}{f} + \frac{y}{h} = 1$, and the equation to

BM, i.e. to a line through $(-g, 0)$ perpendicular to AB, is

$$-\frac{x}{h}+\frac{y}{g}=\frac{g}{h} \text{ (Art. 14)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation to AB, since the intercepts of AB are $-g$, h, is

$$-\frac{x}{g}+\frac{y}{h}=1.$$

The equation to CN, that is, to a line through $(f, 0)$ perpendicular to AB is

$$\frac{x}{h}+\frac{y}{g}=\frac{f}{h}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The point where BM, CN meet is found by combining (1) and (2).

It is the point $x=0$, $y=\dfrac{fg}{h}$.

And this point lies on the line $x=0$, that is, on AL. Thus AL, BM, CN meet in a point.

Call this point the *Orthocentre* of the triangle ABC, or of the points A, B, C; and let it be D. Then of the four points A, B, C, D, any one is the orthocentre of the other three.

(2) ABC *is a triangle right-angled at* C. CD *is at right angles to* AB, CE *bisects the angle* ACB, *and* CF *bisects* AB *in* F. *A line from* B *at right angles to* CF *meets* CD *in* α, *and* Bβ *is at right angles to* CE *and meets* CE *in* β. *Prove that the points* α, β, F *lie in a straight line.*

Take CA, CB for axes of x and y, and let $CA=a$, $CB=b$. Then the equations to CD, CE, CF are

$$ax=by \ (1), \quad x=y \ (2), \quad \frac{x}{a}=\frac{y}{b} \ (3),$$

the first being obtained from the equation to AB (Art. 45).

Also the equation to $B\alpha$ is $ax+by=b^2$ (4), and the equation to $B\beta$ is $x+y=b$ (5).

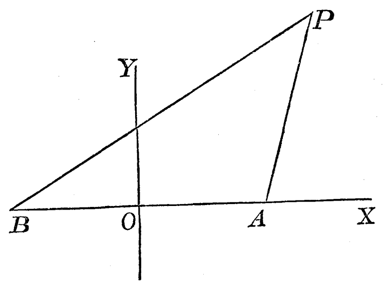
The co-ordinates of α, found by combining (1) and (4), are

$\frac{b^2}{2a}, \frac{b}{2}$. Those of β, found by combining (2) and (5), are $\frac{b}{2}, \frac{b}{2}$.

Therefore the equation to $\alpha\beta$, since α and β have the same y is $y = \frac{b}{2}$.

This equation is satisfied by the co-ordinates of F. That is, F lies on the line $\alpha\beta$.

(3) *A point moves so that the difference of the squares of its distances from two given points is constant. Find the locus of the point.*



Take the line joining the two given points A, B for the axis of x, and the middle point of AB for origin. Let $OA = a$, and let xy be the moving point P. Then

$$PA^2 = (x-a)^2 + y^2, \quad \text{and} \quad PB^2 = (x+a)^2 + y^2.$$

If then $PB^2 - PA^2 =$ a constant quantity k^2,

$$4ax = k^2 \text{ and } x = \frac{k^2}{4a}.$$

This is the equation to P's locus, and thus P moves in a straight line perpendicular to AB and distant $\frac{k^2}{4a}$ from the middle point of AB.

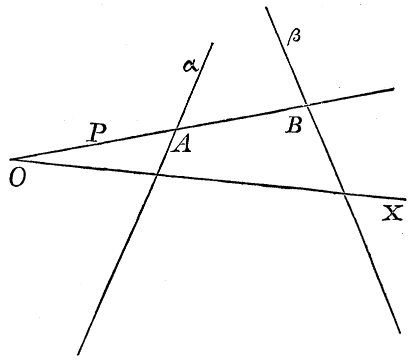
We have confined ourselves to the case in which $PB > PA$. Strictly speaking, the locus is two straight lines which coincide when k is zero.

(4) O *is a fixed point and* OAB *is any line through* O *meeting in* A, B *two given lines* Aα, Bβ.

In OAB *a point* P *is taken such that*

$$\frac{1}{\mathrm{OP}} = \frac{1}{\mathrm{OA}} + \frac{1}{\mathrm{OB}}.$$

What is the locus of P*?*



Take O for pole and draw an initial line OX. Let the equations to $A\alpha$, $B\beta$ be

$$\frac{1}{r} = a\cos\theta + b\sin\theta, \quad \frac{1}{r} = a'\cos\theta + b'\sin\theta,$$

(these are perfectly general forms).

If the angle XOP be θ, then $\frac{1}{OA} = a\cos\theta + b\sin\theta$, and

$$\frac{1}{OB} = a'\cos\theta + b'\sin\theta;$$

and $$\frac{1}{OP} = \frac{1}{OA} + \frac{1}{OB} = (a+a')\cos\theta + (b+b')\sin\theta.$$

That is, at P, $\frac{1}{r} = (a+a')\cos\theta + (b+b')\sin\theta$.

Thus the locus of P is a straight line.

Examples on Chapter III.

Arts. 32—38.

1. Find the point of intersection of the lines

$$3x + 5y = 4, \ \ y = 7x + 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (\omega).$$

2. The lines

$$2x - 3y + 5, \ \ x - y + 4, \ \ 5x - 6y + 20,$$

meet in a point.. (ω).

3. Prove that the equation $y - k = m(x - h)$ can by giving a proper value to m be made the equation to any straight line through (h, k) .... (ω).

4. Find the equations to the straight lines through the point $(1, -2)$, whose acute angles of inclination to Ox are each 45°.

5. Find the equation to a line through the point (h, k) parallel to the line $y = mx$ .... (ω).

6. Find the equation to the line which joins the origin to the intersection of the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{b} + \frac{y}{a} = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (\omega).$$

7. Find the equations to the lines which pass through the following pairs of points:

(1) (a, b), $(-a, -b)$, (2) $(2, -1)$, $(-3, 0)$,
(3) $(4, 7)$, $(4, 1)$, (4) $(0, a)$, $(0, -a)$...(ω).

Arts. 39—50.

8. Find the distances of the lines

$$x + y + 6 = 0, \ \ x + y - 6 = 0$$

from the origin, and find the distance between the lines.

9. Find the angle between the lines

$$y = 2x, \ \ y = 3x + 4,$$

and the distances of these lines from the point $(3, 6)$.

10. Apply Art. 33 to find the condition of parallelism of the lines $Ax + By + C$, $A'x + B'y + C'$.

11. Find the equations to two lines through the point $(0, -1)$, one parallel and the other perpendicular to the line $2x - 3y + 5$.

12. Find the equation to a line through the origin perpendicular to the line joining the points (a, b), (c, d). Find also the length of this perpendicular.

13. Apply Arts. 9 and 46 to find the area of a triangle whose vertices are given.

14. The axes being inclined at 120^0, find the angle between the lines $x + 2y = 0$, $x - y = 0$, and the equation to a line through the point $(1, -1)$ perpendicular to the line $y + 3x = 2$.

15. The distance of the point $x'y'$ from the line $Ax' + By' + C$ is
$$\frac{(Ax' + By' + C)\sin\omega}{\sqrt{A^2 - 2AB\cos\omega + B^2}}.$$

16. The condition that the lines
$$Ax + By + C,\ A'x + B'y + C,$$
may be equally inclined to the axis of x in opposite directions is
$$\frac{B}{A} + \frac{B'}{A'} = 2\cos\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\omega).$$

17. Find the equations to the bisectors of the angles between the axes, and shew that these bisectors are at right angles...(ω).

18. Deduce the result of Art. 49 from that of Art. 38.

Arts. 50—52.

19. Transform the equation $x^2 + y^2 = 2cx$, by changing the origin to the point $(c, 0)$.................................... (ω).

20. Transform $x^2 - y^2 = a^2$ by turning the axes through 45^0.

21. Transform $r^2 = a^2 \cos 2\theta$ to rectangular co-ordinates.

22. If the origin be transferred to the point h, k, and the axes be then turned through an angle θ, the formulæ of transformation will be

$$x = x' \cos\theta - y' \sin\theta + h, \quad y = x' \sin\theta + y' \cos\theta + k.$$

23. The degree of an equation cannot be altered by any transformation in rectangular co-ordinates. Could it be altered by any transformation in oblique co-ordinates?

24. Transform the equation $y^2 = lx$, so that the axis of x may remain unchanged while that of y is brought nearer to it by 30 degrees.

25. Obtain formulæ for transforming from a system of rectangular co-ordinates to a system which has the lines

$$y = x \tan\alpha, \quad y = x \tan\beta$$

for axes.

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26. Find the area of the triangle formed by the three lines

$$y = 0, \quad y + x = a, \quad y = mx + c.$$

27. Why is the condition in Ex. 16 free from C and C'? What is $\dfrac{A}{B} + \dfrac{A'}{B'} = 2\cos\omega$ the condition for?

28. Find the equation to a line perpendicular to the axis of x through the point (h, k) .... (ω).

29. In the figure to Euclid, I. 47, the lines FC, KB, AL meet in a point.

30. Prove by taking two sides for axes that the lines drawn from the angles of a triangle to the middle points of the opposite sides meet in a point.

31. A point moves so that the sum of the squares of its distances from two points x_1y_1, x_2y_2 is equal to the sum of the squares of its distances from two other points x_3y_3, x_4y_4. Prove that the locus of the point is the straight line

$$2x(x_1 + x_2 - x_3 - x_4) + 2y(y_1 + y_2 - y_3 - y_4)$$
$$= x_1^2 + x_2^2 - x_3^2 - x_4^2 + y_1^2 + y_2^2 - y_3^2 - y_4^2.$$

32. A point moves so that the sum of the squares of its distances from n given points = the sum of the squares of its distances from n other given points. Find the locus of the point.

33. A line OP meets n fixed lines in $A_1, A_2 \dots A_n$, and is drawn from a fixed point O. If $\frac{1}{OP} = \frac{1}{OA_1} + \frac{1}{OA_2} + \dots \frac{1}{OA_n}$, the locus of P is a straight line.

34. If the equation to a straight line involve some unknown constant in the first degree, and all the other constants be known, the line passes through a known point........................(ω).

35. Through the intersection of $y - 2x = 1$, $x + 5y = 6$, draw a line perpendicular to $2y + 4x = 7$.

36. Upon the sides of a triangle as diagonals parallelograms are described, having their sides parallel to two given lines. Prove that the other diagonals of the parallelograms meet in a point.. (ω).

37. Find the equations to the straight lines through the point (1, 3) which are inclined at 30° to the line $x + y\sqrt{3} = 2$.

38. If AL, BM, CN be the perpendiculars from A, B, C, on BC, CA, AB and MN, NL, LM meet BC, CA, AB respectively in F, G, H, then F, G, H lie in a straight line.

39. A straight line moves so that the sum of the reciprocal of its intercepts is constant. Prove that it passes through a fixed point.

CHAPTER IV.

THE STRAIGHT LINE.

54. We now return to the subject of Art. 21.

The equation $x^2 - y^2 = 0$, or $(x-y)(x+y) = 0$ (1), is satisfied by the co-ordinates of any point whose $x - y$ vanishes, and by the co-ordinates of any point whose $x + y$ vanishes: and if neither the $x - y$ nor the $x + y$ of a given point vanishes, the co-ordinates of that point do *not* satisfy the equation (1).

That is to say, the equation (1) is satisfied by the co-ordinates of any point which lies in one of the straight lines $x - y$, $x + y$, and by no other point.

Equation (1) represents therefore two straight lines, namely the lines $x - y$, $x + y$.

In like manner the equation $(x - 2y + 1)(2x + 3y + 2) = 0$, represents two straight lines, namely, the lines $x - 2y + 1$, $2x + 3y + 2$.

The equation $(x - y + a)(x - y) = 0$ represents two parallel straight lines of which one approaches nearer and nearer to the other as a is made more and more nearly equal to zero.

If a be zero, these lines are coincident. Thus $(x - y)^2 = 0$ represents two coincident straight lines.

The equation $(x - 2y)(2x - y + 3)(x - 2y + c) = 0$ represents *three* straight lines, namely, $x - 2y$, $2x - y + 3$, and $x - 2y + c$, of which two are parallel. The equation

$$(x - 2y)^2 (2x - y + 3) = 0$$

represents three straight lines, of which two are coincident.

The equation $y^2 - 5xy + by^2 = 0$ represents two straight lines, $y = 2x$, $y = 3x$, passing through the origin.

The equation $Ax^2 + Bxy + Cy^2 = 0$ represents two straight lines passing through the origin: for if α, β be the roots of the equation

$$Cz^2 + Bz + A = 0,$$

then $$Cz^2 + Bz + A = C(z-\alpha)(z-\beta) \text{ identically:}$$

therefore $$Ax^2 + Bxy + Cy^2 = x^2\left\{C\left(\frac{y}{x}\right)^2 + B\left(\frac{y}{x}\right) + A\right\}$$

$$= Cx^2\left(\frac{y}{x} - \alpha\right)\left(\frac{y}{x} - \beta\right)$$

$$= C(y - \alpha x)(y - \beta x).$$

Thus the two straight lines are $y = \alpha x$, $y = \beta x$. The lines coincide if $\alpha = \beta$, i.e. if $B^2 - 4AC = 0$.

And, generally, any homogeneous equation represents a series of straight lines through the origin. For

$$A_0 y^n + A_1 y^{n-1}x + A_2 y^{n-2}x^2 + \ldots + A_{n-1}yx^{n-1} + A_n x^n = 0 \ldots (1)$$

is the general type of such equations, n being a positive integer. Now if α_1, $\alpha_2 \ldots \alpha_n$ be the roots of

$$A_0 z^n + A_1 z^{n-1} + \ldots + A_{n-1}z + A_n = 0 \ldots\ldots\ldots\ldots (2),$$

then the first side of (2)

$$= A_0(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n) \text{ identically.}$$

Thus the first side of (1)

$$= A_0 x^n \left(\frac{y}{x} - \alpha_1\right)\left(\frac{y}{x} - \alpha_2\right) \ldots\ldots \left(\frac{y}{x} - \alpha_n\right)$$

$$= A_0 (y - \alpha_1 x)(y - \alpha_2 x) \ldots\ldots (y - \alpha_n x).$$

Therefore the locus of (1) consists of n straight lines passing through the origin.

Ex. $y^3 - 3x^2y + 2x^3 = 0$ represents three straight lines

$$y + 2x, \quad y - x, \quad y - x,$$

of which two are coincident.

The equation $8x^2 - 26x + 21 = 0$ represents two straight lines parallel to the axis of y:

$$2x = 3, \quad 4x = 7.$$

If $f(x)$ denote $A_0x^n + A_1x^{n-1} + \ldots + A_n$, so that

$$f(x) = A_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n),$$

then $f(x) = 0$ represents n straight lines parallel to the axis of y.

Similarly $f(y) = 0$ represents n straight lines parallel to the axis of x.

The equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2)\ldots(A_nx + B_ny + C_n) = 0 \quad (3)$$

represents n straight lines, viz. those corresponding to the several factors.

If $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$, i. e. if $(A_1x + B_1y + C_1)(A_2x + B_2y + C_2)$ be a perfect square, then two of the n straight lines coincide. If a perfect r^{th} power occur in the product of the n linear factors, there correspond r coincident lines in the locus.

If $C_1, C_2 \ldots C_n$ all vanish, equation (3) is homogeneous, and all the lines pass through the origin.

If $A_1, A_2 \ldots A_n$ all vanish, the lines are all parallel to the axis of y. If $B_1, B_2 \ldots B_n$ all vanish, the lines are all parallel to the axis of x.

All these remarks apply equally to oblique and rectangular co-ordinates. The polar equation to a series of lines through the origin is the same as the Cartesian, but has l and m for x and y. (Art. 52.)

55. *To find the angle between the straight lines*

$$\mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 = 0,$$

the axes being rectangular.

Let m_1, m_2 be the values of z in the equation

$$Cz^2 + Bz + A = 0,$$

so that
$$m_1 + m_2 = -\frac{B}{C}, \quad m_1 m_2 = \frac{A}{C}.$$

Then the tangent of the angle

$$= \frac{m_1 \sim m_2}{1 + m_1 m_2} = \frac{\sqrt{B^2 - 4AC}}{A + C}.$$

The straight lines coincide if $B^2 - 4AC = 0$ (in which case $Ax^2 + Bxy + Cy^2$ is a perfect square), and they are at right angles if $A + C = 0$, i. e. if the coefficients of x^2 and y^2 be equal and of opposite signs. Thus the lines $2x^2 - 3xy - 2y^2 = 0$ are at right angles.

56. We return to oblique co-ordinates. We know that the equation $4x - 5y + 6 = 0$ (1) represents a straight line, which we may call 'the line $4x - 5y - 6$.' This line may with equal propriety be called 'the line $-4x + 5y + 6$.' Every straight line, in fact, has two names. The lines $Ax + By + C$ is also the line $-Ax - By - C$: the line x is also the line $-x$. Can any use be made of this duality?

The $4x - 5y - 6$ of any point which is not on the line (1), does not vanish: thus the $4x - 5y - 6$ of the origin is -6: the $4x - 5y - 6$ of a point on the axis of x at a very great distance is positive. This point is on the opposite side of the line from the origin. The $4x - 5y - 6$ of *any* point on the same side of the line as the origin is negative, and the $4x - 5y - 6$ of any point on the other side of the line is positive.

In like manner the $Ax + By + C$'s of any two points lying on opposite sides of the line $Ax + By + C$ are of opposite signs. For, in the first place, the $Ax + By + C$ of a point cannot change its sign without passing through the value zero, that is, the $Ax + By + C$ of a point is the same so long as the point remains on one side of the line. It only remains to prove that the signs on opposite sides of the line are not the same. To the reader of

the Differential Calculus this will be evident, since $Ax+By+C$ cannot, being linear, have a maximum or minimum value. But we may also satisfy ourselves by taking severally the special cases which arise by giving various signs to A, B, C. For instance, let A, B, C be all positive. Then the line crosses the axes in the third quadrant, and the $Ax+By+C$ of the origin is positive, while that of the point $(-\infty, 0)$, a point on the opposite side of the line, is negative. Or again, suppose A positive, B negative, and C zero. Then the line passes through the origin and lies in the first and third compartments. We cannot now use the origin for a testing point, but we may take two such points as $(1, 0)$, $(-1, 0)$, which lie on opposite sides of the line. The $Ax+By$ of the one is A, and that of the other $-A$. Similarly other cases may be treated. Thus every straight line has two sides, a positive and a negative. *Which* shall be positive we determine by the manner in which we name the line. The origin is on the positive side of the line $x-2y+1$, but on the negative side of the line $-x+2y-1$.

This throws light on the occurrence of the radical in the expression for the length of the perpendicular from the point xy on the line $Ax+By+C$: and we see that when we have once fixed on the name $Ax+By+C$ of a line, then the $Ax+By+C$ of any point varies as the distance of the point from the line, for the only variable part of the expression aforesaid is $Ax+By+C$. Thus the x of any point varies as the distance of the point from a certain line, being actually that distance in the case of rectangular co-ordinates. The $x\cos\alpha+y\sin\alpha-p$ of any point is the distance of that point from the line

$$x\cos\alpha+y\sin\alpha-p.$$

57. A point is equidistant from the lines

$$x\cos\alpha+y\sin\alpha-p,\quad x\cos\beta+y\sin\beta-q,$$

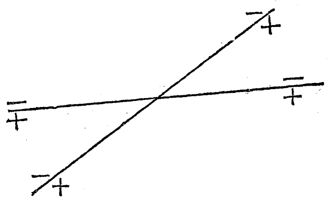
if its $x\cos\alpha+y\sin\alpha-p$ and $x\cos\beta+y\sin\beta-q$ be numerically equal: that is to say, the equations

$$x\cos\alpha+y\sin\alpha-p=x\cos\beta+y\sin\beta-q\;\ldots\ldots\ldots(1)$$
$$x\cos\alpha+y\sin\alpha-p=-(x\cos\beta+y\sin\beta-q)\ldots\ldots(2)$$

represent the bisectors of the angles between the straight lines

$$x \cos \alpha + y \sin \alpha - p \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$x \cos \beta + y \sin \beta - q \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$



The lines (3), (4) have each a positive and a negative side. Thus they divide the plane of operations into four compartments, of which one is entirely on the positive side of both lines, and another, namely the vertically opposite, entirely on the negative side of both lines. In these two compartments lies the bisector (1). In the other two lies the bisector (2).

Ex. To find the equations to the bisectors of the angles between the lines

$$2x + y = 1, \quad y + 3x = 2.$$

Expressed in the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

the given equations are

$$\frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0, \qquad \frac{y}{\sqrt{10}} + \frac{3x}{\sqrt{10}} - \frac{2}{\sqrt{10}} = 0.$$

Thus the required equations are

$$(2\sqrt{2} + 3)\, x + (\sqrt{2} + 1)\, y - (\sqrt{2} + 2) = 0,$$

and $$(2\sqrt{2} - 3)\, x + (\sqrt{2} - 1)\, y + 2 - \sqrt{2} = 0.$$

58. *To find the equation to the bisectors of the angle between the lines*

$$Ax^2 + Bxy + Cy^2 = 0.$$

Let the given lines be $y = m_1 x$, $y = m_2 x$, so that m_1, m_2 are the roots of $Cz^2 + Bz + A = 0$.

The bisectors of the angles between $y = m_1 x$, $y = m_2 x$ are

$$\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \frac{y - m_2 x}{\sqrt{1 + m_2^2}} \text{ and } \frac{y - m_1 x}{\sqrt{1 + m_1^2}} = -\frac{y - m_2 x}{\sqrt{1 + m_2^2}}.$$

These lines are expressed by the single equation

$$\frac{(y - m_1 x)^2}{1 + m_1^2} - \frac{(y - m_2 x)^2}{1 + m_2^2} = 0,$$

or $$y^2 (m_2 + m_1) + 2xy (1 - m_1 m_2) - x^2 (m_2 + m_1) = 0,$$

or $$y^2 + 2xy \cdot \frac{C - A}{B} - x^2 = 0.$$

We see by Art. 55 that these straight lines are at right angles.

59. *Given that the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ ....(1)}$$

represents two straight lines, to find the angle between them.

Let hk be the point of intersection of the two lines. Transfer the origin to hk, by writing $x + h$, $y + k$ for x, y. The new equation is

$$Ax^2 + Bxy + Cy^2 = 0,$$

for, as both the lines pass through the new origin, the terms of one dimension and the constant term disappear.

The angle between the lines is therefore $\tan^{-1} \frac{\sqrt{B^2 - 4AC}}{A + C}$.

Thus the angle between the lines represented by equation (1) is the same as the angle between the lines

$$Ax^2 + Bxy + Cy^2 = 0 \text{ ....(2)},$$

i.e. the same as the angle between the lines represented by the terms of highest dimensions in the equation (1). And in fact the lines (1) are *parallel* to the lines (2).

60. In discussing the equation

$$Ax^2 + Bxy + Cy^2 = 0,$$

we have supposed the first side to be the product of two *real* linear factors. If, however, B^2 be less than $4AC$ the factors will be imaginary, and the locus consists of two imaginary straight lines, viz.

$$y + \frac{B - \sqrt{-(4AC - B^2)}}{2A} \cdot x = 0 \text{ and } y + \frac{B + \sqrt{-(4AC - B^2)}}{2A} \cdot x = 0.$$

These imaginary straight lines intersect in a real point, viz. the origin: for their equations are each satisfied by making

$$x = 0, \quad y = 0.$$

Ex. $x^2 + y^2 = 0$ represents the two imaginary straight lines

$$x + y\sqrt{-1} = 0, \quad x - y\sqrt{-1} = 0,$$

which intersect in a real point, viz. the origin.

61. In what cases does the expression

$$ax^2 + bxy + cy^2 + dx + ey + f \dots\dots\dots\dots\dots (1)$$

break up into two factors? For instance, how are we to know whether

$$2x^2 - 13xy + 6y^2 - 3x - 2y + 6$$

does or does not so break up? The expression (1) includes all those varieties of the rational and integral function of variables which are of the second degree, but it is convenient to replace it by an equally general form

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy \dots\dots\dots\dots (2)$$

Now if (2) be of the form

$$(Ax + By + C)(A'x + B'y + C'),$$

then by equating (2) to zero, and solving the equation for y, we ought to find two rational roots, $-\frac{Ax + C}{B}$, $-\frac{A'x + C'}{B'}$; rational, that is to say, as far as x is concerned, for A, B, C

may be surds or even imaginary. Now that the equation

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0$$

may give two such rational values for y, it is necessary and sufficient that

$$(a' + c'x)^2 - b\,(ax^2 + c + 2b'x)$$

be a perfect square: that is, that

$$(a'c' - bb')^2 - (c'^2 - ab)\,(a'^2 - bc) \text{ may} = 0.$$

Thus if

$$(a'c' - bb')^2 = (c'^2 - ab)\,(a'^2 - bc),$$

or, which is the same thing, if

$$abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2 = 0 \quad \ldots\ldots\ldots(3),$$

then

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy \quad \ldots\ldots\ldots(2)$$

breaks up into two linear factors.

We have supposed b not zero. If $b = 0$, then, if a be not zero, solution with regard to x leads to the same result. If *both* a and $b = 0$, then we may reason thus: The expression (2) becomes

$$y\,(2c'x + 2a') + (2b'x + c),$$

and this cannot be the product of two linear factors unless $2c'x + 2a'$ be an exact multiple of $2b'x + c$. This requires that $\frac{c'}{b'}$ shall $= \frac{2a'}{c}$, or $2a'b' = cc'$, which is indeed what (3) tells us, since c' is not zero. Hence (3) is universally applicable.

It is also the condition under which

$$ax^2 + by^2 + cz^2 - 2a'yz - 2b'zx - 2c'xy \ldots\ldots\ldots\ldots\ldots(4)$$

breaks up into two factors of the form

$$Ax + By + Cz,\; A'x + B'y + C'z,$$

and the expression on the left-hand side of (3) is called the *Discriminant* of the expression (4).

Ex. What value must we give to λ in order that

$$\lambda x^2 + xy + y^2 - 2x - 3y + 1$$

may break up into two factors?

Here $\quad a = \lambda,\ b = 1,\ c = 1,\ a' = -\frac{3}{2},\ b' = -1,\ c' = \frac{1}{2},$

thus (3) becomes

$$\lambda + \frac{3}{2} - \frac{9}{4}\lambda - 1 - \frac{1}{4} = 0, \text{ or } \lambda = \frac{1}{5}.$$

Examples on Chapter IV.

1. What are the loci of the following equations?

(1) $x^2 - a^2 = 0,$ (2) $(x-a)^2 + (y-b)^2 = 0,$

(3) $(x-a)(y-b) = 0,$ (4) $y^2 + 4xy^2 + 3x^2 = 0,$

(5) $y^2 + 4xy + 4y^2 = 0,$ (6) $x^3 - a^3 = 0 \ldots\ldots (\omega)$

2. Find the condition that one of the lines

$$ax^2 + 2bxy + cy^2 = 0,$$

may coincide with one of the lines

$$a'x^2 + 2b'xy + c'y^2 = 0 \ldots\ldots (\omega)$$

3. The lines $4y^2 + \sqrt{11}\,.\,xy = x^2$ include an angle of 60^0.

4. Prove that the points (1, 2), (3,–4), lie on opposite sides of the line $3x - 2y = 1$, and on the same side of the line $3x = y$.

5. Two lines are drawn in directions $[l, m,]\ [l', m']$. Prove that the bisectors of the angles between them are in directions

$$(l + l',\ m + m'),\ (l - l',\ m - m').$$

6. Find the bisectors of the angles between $Ax + By + C$, and $A'x + B'y + C'$.

7. Given that $ax^2+by^2+c+2a'y+2b'x+2c'xy=0$, represents two straight lines, prove that at their point of intersection $ax+c'y+b'=0$ and $c'x+by+a'=0$. (See Art. 50.)

8. Prove that the equation in Ex. 7 represents two straight lines if the equations

$$ax+c'y+b'z=0,\quad c'x+by+a'z=0,$$

and
$$b'x+a'y+cz=0,$$

can exist together for other values of x, y, z than 0, 0, 0.

9. Does $x^2-4xy+4y^2+6x-12y+9=0$, represent two straight lines?......(ω)

10. What value must be given to f in order that

$$x^2+fy^2+3xy+x+y=1,$$

may represent two straight lines? (ω)

11. What values must be given to f and g in order that

$$x^2+fy^2+3xy+x+gy=1,$$

may represent two straight lines intersecting in the axis of x? (ω)

12. Find the area of the parallelogram whose sides are

$$Ax+By+C,\ Ax+By+C',\ ax+by+c,\ ax+by+c',$$

and the equations to the diagonals.

13. Interpret the equation $\sin 3\theta=0$.

14. Find the equation to a line passing through a given point and dividing the line joining two other points in a given ratio.

The angular points of a triangle are x_1y_1, x_2y_2, x_3y_3. Prove that the bisector of the angle at $x_1\,y_1$ is represented by

$$\frac{x-x_1}{y-y_1}$$
$$=\frac{(x_2-x_1)\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+(x_3-x_1)\sqrt{(x_2-x_1)^2+(y_2-y_1)}}{(y_2-y_1)\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+(y_3-y_1)\sqrt{(x_2-x_1)^2+(y_2-y_1)}}$$

15. Find the ratio in which the line joining x_1y_1, x_2y_2, is cut by the line $Ax+By+C$. (ω)

16. A line is drawn from the point $x'\ y'$ in direction $[l, m]$, to meet the line $Ax+By+C$. Prove that its length is

$$-\frac{Ax'+By'+C}{Al+Bm}. \quad (\omega)$$

17. A line is drawn in a given direction and is terminated by two given lines. Find the locus of a point which divides it in a given ratio. (ω)

18. Prove by a figure that the distance of $x'y'$ from the line $x\cos\alpha+y\cos\beta=p$, is $x\cos\alpha+y\cos\beta-p$,

or $$p-x\cos\alpha-y\cos\beta,$$

according as $x'y'$ is or is not on the origin-side of the line.

19. PAB, PCD are triangles on given bases AB, CD, and the sum of their areas is constant. Find the locus of P. Is this locus infinite?

20. One vertex of a rigid triangle is fixed, and another moves along a given straight line. Find the locus of the third.

CHAPTER V.

ABRIDGED NOTATION OF THE STRAIGHT LINE.

62. LET the symbol L denote the expression $Ax + By + C$. Then $L = 0$ is the equation to a straight line, and we may call this line 'the line L'. If M denote $A'x + B'y + C'$, $M = 0$ is the equation to 'the straight line M', and $L = \lambda M$, by giving the right value to λ, can be made to represent any straight line passing through the point where the lines L and M intersect (Art. 37). Thus if we wish for the equation to the straight line joining that point to the point (h, k) :—let L_1, M_1, be the special values of L, M obtained by substituting h, k for x, y: then the equation required is $\dfrac{L}{L_1} = \dfrac{M}{M_1}$.

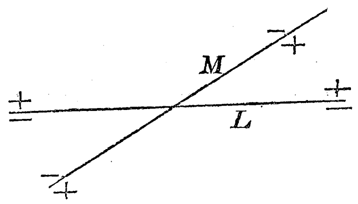
The L of any point is proportional to the distance of that point from the line L, being in fact equal to the length of that distance

$$\times \frac{\sqrt{A^2 - 2AB\cos\omega + B^2}}{\sin\omega} \text{ (Chap. III. Ex. 15).}$$

So long as the L of a point remains the same, the point is on the same straight line parallel to the line L. The equation to any straight line parallel to the line L is expressible in the form $L =$ constant. If the L's of two points have the same sign the points lie on the same side of the line L, and if their L's have different signs the points lie on opposite sides of the line L. The lines $L = c$, $L = -c$ are parallel to the line L, on opposite sides of it, and equidistant from it.

The lines L, M divide the plane of operations into 4 compartments. The lines $L - \lambda M$ lies in two of these and the line $L + \lambda M$ in the other two. For let P be a point in the line

$L-\lambda M$, and Q a point in the line $L+\lambda M$, and let P, Q be on the same side of the line M: then the M's of P and Q have the same signs, and therefore, as appears from the equations $L=\lambda M$, $L=-\lambda M$, the L's of P and Q have opposite signs: that is, P and Q are on opposite sides of the line L. If λ be positive, the line $L-\lambda M$ lies in the $++$ compartment and in the $--$ compartment; and the line $L+\lambda M$ lies in the $+-$ compartment and the $-+$ compartment: for at any point in



the first line the L and M have the same sign, since their ratio is the positive quantity λ: and at any point in the other line the L and M have opposite signs, since their ratio is the negative quantity $-\lambda$.

63. If L, M, N be three given lines not meeting in a point, then the equation $lL+mM+nN=0$ (1) can, by giving values to l, m, n which shall be in a proper proportion, be made to represent any straight line whatever. For let the proposed straight line be that which joins the points x_1y_1, x_2y_2. Let L_1, M_1, N_1 be the values which L, M, N take when x_1, y_1 are substituted for x, y; in other words, let L_1, M_1, N_1 be the L, M, N of x_1y_1: and let L_2, M_2, N_2 be the L, M, N of x_2y_2. Then, for determining the ratios of l, m, n we have the equations $L_1l+M_1m+N_1n=0$, $L_2l+M_2m+N_2n=0$: whence

$$l:m:n=M_2N_1-M_1N_2:N_2L_1-L_1N_2:\ L_2M_1-L_1M_2.$$

Thus the equation sought for is

$$(M_2N_1-M_1N_2)\,L+(N_2L_1-L_1N_2)\,M+(L_2M_1-L_1M_2)N=0\ldots(2)$$

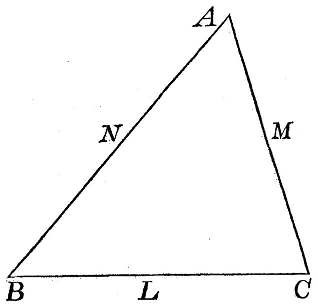
If the straight lines L, M, N meet in a point, N is of the form $\lambda L+\mu M$ (λ, μ being constants) and equation (2) becomes an identity. The form $lL+mM+nN=0$ can in this case only

represent lines passing through the point of intersection of the lines L, M, N, being equivalent to

$$(l+\lambda n)\,L+(m+\mu n)\,M=0.$$

64. As an example of the method of *abridged notation* we shall now solve the following problem.

On the sides BC, CA, AB of a triangle ABC pairs of points are taken, B_1, C_1; C_2, A_2; A_3, B_3, such that the points of intersection of BC with B_3C_2, of CA with C_1A_3, and of AB with A_2B_1 lie in a straight line. BC_2, CB_3 intersect in P, CA_3 in Q, and AB_1, BA_2 in R. Prove that AP, BQ, CR meet in a point.



Let $L=0$, $M=0$, $N=0$ be the equations to BC, CA, AB, and let $M=\lambda N$, $M=\lambda' N$, $N=\mu L$, $N=\mu' L$, $L=\nu M$, $L=\nu' M$ denote AB_1, AC_1, BC_2, BA_2, CA_3, CB_3 respectively.

Then B_3C_2 joins the intersection of N, $L-\nu' N$ to the intersection of M, $N-\mu L$. Suppose its equation, then, to be $kN+L-\nu' M=0$ or $k'M+N-\mu L=0$. (Art. 37.)

From the identity of the two forms we get

$$-\frac{1}{\mu}=-\frac{\nu'}{k'}=k.$$

Thus the equation to B_3C_2 is

$$-\frac{N}{\mu}+L-\nu' M=0,$$

or
$$L=\nu' M+\frac{N}{\mu}.$$

Similarly the equations to C_1A_3, A_2B_1 are

$$M = \lambda' N + \frac{L}{\nu}, \qquad N = \mu' L + \frac{M}{\lambda}.$$

Now let $lL + mM + nN$ be the line on which lie the intersections of B_3C_2, C_1A_3, A_2B_1 with the corresponding sides of the triangle. Then $lL + mM + nN = 0$ is satisfied by those values of x and y which make $L = 0$ and $L = \nu' M + \frac{N}{\mu}$, i.e. which make $L = 0$ and $M = -\frac{N}{\mu\nu}$; therefore

$$m - n\mu\nu' = 0,$$

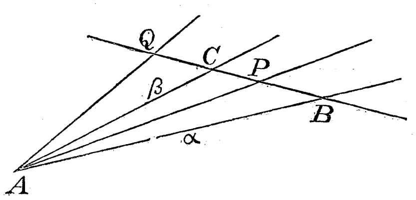
similarly $$n - l\nu\lambda' = 0,$$

and $$l - m\lambda\mu' = 0,$$

and therefore $$\lambda\mu\nu\,\lambda'\mu'\nu' = 1.$$

Again; at P we have $N = \mu L$, $L = \nu' M$: therefore P lies on the line $N = \mu\nu' M$. Therefore, as this line passes through A, it is the line AP. Similarly $L = \nu\lambda' N$, $M = \lambda\mu' L$ are the lines BQ, CR. And as $\lambda\mu\nu\lambda'\mu'\nu' = 1$, these three equations can coexist, that is, AL, BM, CN meet in a point.

65. Let the abbreviated form be $p - x \cos\alpha - y \sin(\omega - \alpha) = 0$, p being positive; and let α stand for $p - x\cos\alpha - y\cos(\omega - \alpha)$. Then all points with a positive α are on the same side of the line as the origin. Let β, in like manner, denote $q - x\cos\beta - y\cos(\omega - \beta)$, q being positive. Then the α and β of any point are its distances from the lines α, β. The lines $\alpha - \beta$, $\alpha + \beta$ are the bisectors of the angles between the lines α, β. The lines $\alpha - \kappa\beta$, $\alpha + \kappa\beta$ divide the angle between α and β internally and externally so that the sines of the segments are as 1 to k.



Thus if AB, AC be the lines α, β, and AP, AQ the lines $\alpha - \kappa\beta$, $\alpha + \kappa\beta$, then $\frac{\sin PAB}{\sin PAC} = \frac{\sin QAB}{\sin QAC}$.

If $BPCQ$ be any *transversal*, or straight line drawn across the *pencil* of lines AB, AP, AC, AQ, then, as the reader can prove, BC is divided harmonically at P and Q, or, in other words, $BPCQ$ is a harmonic range. The pencil of lines emanating from A is called a *harmonic* pencil.

In general, if AB, AP, AC, AQ be any four fixed straight lines which are met by a variable straight line in B, P, C, Q, the *anharmonic ratio* of the range $BPCQ$ is $\frac{BP}{CP} \div \frac{BQ}{CQ}$ (being the ratio of the ratios in which BC is divided in the points P and Q), and this ratio is constant, being equal to

$$\frac{\sin BAP}{\sin CAP} : \frac{\sin BAQ}{\sin CAQ}.$$

66. It is usual to reserve Greek letters for that form of equation which expresses a straight line in terms of the distance from the origin, and the angular position of that distance.

67. The lines $L - kM$, $L + kM$ form with the lines L, M a harmonic pencil; for, if α, β be altered forms of L and M, $L = \alpha\sqrt{A^2 + B^2}$, and $M = \beta\sqrt{A'^2 + B'^2}$, so that the equations $L - kM = 0$, $L + kM = 0$ become $\alpha - k\sqrt{\frac{A'^2+B'^2}{A^2+B^2}} . \beta = 0$, and $\alpha + k\sqrt{\frac{A'^2 + B'^2}{A^2 + B^2}} . M = 0$, which are of the form $\alpha - k\beta = 0$, $\alpha + k\beta = 0$. It is not necessarily true that the lines $L - M$, $L + M$ are the bisectors of the angles between L and M. They, however, form with L and M a harmonic pencil.

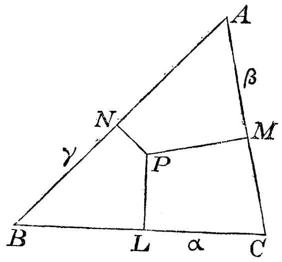
The line $L - kM$ makes with L and M angles equal to those which the line $kL - M$ makes with M, L, and the lines $L - kM$, $L - kM$ lie in the same compartments.

Ex. The line $y = 3x$ makes with Ox an angle equal to

that which $3y = x$ makes with Oy, the axes being rectangular or oblique.

68. Let ABC be a triangle, and let a, b, c be the lengths of the sides BC, CA, AB. Also let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to these sides, the origin of coordinates being within the triangle. Then the α, β, and γ of any point P within the triangle are all positive. If PL, PM, PN be drawn perpendicular to the sides,

$$PL = \alpha, \quad PM = \beta, \quad PN = \gamma.$$



Hence the areas of the triangles BPC, CPA, APB are $\frac{1}{2}a\alpha$, $\frac{1}{2}b\beta$, $\frac{1}{2}c\gamma$, and if Δ denote the area of the triangle ABC,

$$a\alpha + b\beta + c\gamma = 2\Delta \quad \text{.................. (1).}$$

This is an invariable algebraic relation connecting the α, β, γ of a point in the plane of the lines α, β, γ. A point outside the triangle has two of the quantities α, β, γ negative if it lie in one of the angles vertically opposite to the angles of the triangle: otherwise it has two of the quantities α, β, γ positive.

69. We might have defined α, β, γ as the distances of a point from the three sides BC, CA, AB of a triangle of reference ABC, regard being paid to algebraic sign, and thus we might establish a system of *Trilinear* Coordinates on a basis independent of the Cartesian system. We shall, however, not abandon the right of passing from one system to the other, but assume, in order to establish results in Trilinear Coordinates, any results we please in Cartesians. We may call the point whose distances from the sides of the triangle of reference are $\alpha\beta\gamma$ 'the point $\alpha\beta\gamma$.' Thus A is the point $\frac{2\Delta}{a}$, 0, 0.

70. The equation to any straight line can be put in the form

$$l\alpha + m\beta + n\gamma = 0 \text{ (see Art. 63).}$$

In fact the equation to *any* locus can be expressed in terms of α, β, γ. For, as

$$p - x\cos\alpha - y\cos(\omega - \alpha) = \alpha,$$

and
$$q - x\cos\beta - y\cos(\omega - \beta) = \beta,$$

(the reader will not be misled by the double use of α and β), we can express x and y in terms of α and β. Thus an equation $f(xy) = 0$ can be converted into $\phi(\alpha\beta) = 0$, and this can be made homogeneous by introducing $\dfrac{a\alpha + b\beta + c\gamma}{2\Delta}$, which $= 1$.

Ex. Let $xy = m^2$ be an equation in Cartesians, and let it be required to get the corresponding equation in Trilinears.

We have
$$x\cos\alpha + y\cos(\omega - \alpha) = p - \alpha,$$
$$x\cos\beta + y\cos(\omega - \beta) = q - \beta,$$

whence

$$x = \frac{(p - \alpha)\cos(\omega - \beta) - (q - \beta)\cos(\omega - \alpha)}{\cos\alpha\cos(\omega - \beta) - \cos(\omega - \alpha)\cos\beta} = R\alpha + T\beta + S \text{ say,}$$

$$y = -\frac{(p - \alpha)\cos\beta - (q - \beta)\cos\alpha}{\cos\alpha\cos(\omega - \beta) - \cos(\omega - \alpha)\cos\beta} = R'\alpha + T'\beta + S' \text{ say.}$$

Thus the non-homogeneous equation is

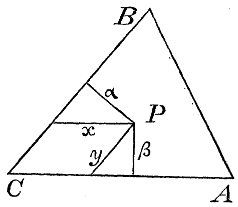
$$(R\alpha + T\beta + S)(R'\alpha + T'\beta + S') = m^2,$$

and the homogeneous equation is

$$(R\alpha + T\beta)(R'\alpha + T'\beta) + \frac{a\alpha + b\beta + c\gamma}{2\Delta}\{S(R'\alpha + T'\beta)$$
$$+ S'(R\alpha + T\beta)\} + \left(\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right)^2(SS' - m^2) = 0.$$

Of course matters would be simplified in this case by supposing two of the sides of the Triangle of Reference to be

coordinate axes. Suppose, for instance, that C is origin, and CA, CB axes of x and y: then $\omega = \angle C$ and $x \sin C = \alpha$, $y \sin C = \beta$.

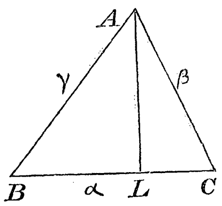


Thus the non-homogeneous equation corresponding to $xy = m^2$ is $\frac{\alpha\beta}{\sin^2 C} = m^2$, and the homogeneous equation is

$$\alpha\beta = \left(\frac{m \sin C}{2\Delta}\right)^2 (a\alpha + b\beta + c\gamma)^2.$$

71. *The three perpendiculars from A, B, C on BC, CA, AB meet in a point.*

Let AL be the perpendicular from A on BC. Then its



equation is of the form $\frac{\beta}{\gamma} = \kappa$. But at L

$$\frac{\beta}{\gamma} = \frac{LC \sin C}{LB \sin B} = \frac{b \cos C \sin C}{c \cos B \sin B} = \frac{\cos C}{\cos B},$$

therefore the equation to AL is $\beta \cos C = \gamma \cos C$. The equations to the other perpendiculars are $\gamma \cos C = \alpha \cos A$, $\alpha \cos A = \beta \cos B$, and these lines meet in the point $\alpha \cos A = \beta \cos B = \gamma \cos C$.

We may observe that any point may be denoted by equations

such as $$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n}.$$

If the actual values of α, β, γ be required, we have

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}=\frac{a\alpha+b\beta+c\gamma}{al+bm+cn}=\frac{2\Delta}{al+bm+cn}.$$

72. It may be shewn by a proof like that used in Art. 28 that any straight line can be expressed by a system of equations of the form

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=r \dots\dots\dots\dots\dots\dots (1).$$

Here α', β', γ' are trilinear coordinates of some fixed point in the line; r is the distance, capable of sign, of the point $\alpha\beta\gamma$ from the point $\alpha'\beta'\gamma'$; and l, m, n are numerically proportional to the sines of the angles which the line makes with BC, CA, AB. As $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ cannot all have the same sign (see a figure), two only of the quantities l, m, n have the same sign. Draw through A, B, C three parallels to the line. One of these crosses the triangle, and to it corresponds the single sign, which is + or − according as we choose the positive direction of r.

Of course the line

$$\frac{\alpha-\alpha''}{l}=\frac{\beta-\beta''}{m}=\frac{\gamma-\gamma''}{n}$$

is parallel to the line (1).

Thus the parallel through C is

$$\frac{\alpha}{l}=\frac{\beta}{m}\left(=\frac{\gamma-\frac{2\Delta}{c}}{n}\right).$$

Suppose we wish to reduce $\alpha-2\beta+3\gamma=0$ (2) to the form (1). We may take for $\alpha'\beta'\gamma'$ the point

$$\alpha=\frac{\beta}{2}=\gamma\left(=\frac{2\Delta}{a+2b+c}\right).$$

A parallel through C must have an equation of the form $\alpha-2\beta+3\gamma=\text{constant}$; let the homogeneous form be

$$\alpha-2\beta+3\gamma=\kappa\,(a\alpha+b\beta+c\gamma) \dots\dots\dots\dots\dots (3).$$

Since the coefficient of l vanishes, $\kappa = \frac{3}{c}$. Thus (3) becomes

$$\alpha - 2\beta = \frac{3}{c}(a\alpha + b\beta).$$

Comparing this with $\frac{\alpha}{l} = \frac{\beta}{m}$ we see that

$$\frac{l}{3b+2c} = \frac{m}{c-3a}.$$

In like manner

$$\frac{l}{3b+2c} = \frac{n}{-2a-b}.$$

Thus the reduced form of (2) is

$$\frac{\alpha - \dfrac{2\Delta}{a+2b+c}}{3b+2c} = \frac{\beta - \dfrac{4\Delta}{a-2b+c}}{c-3a} = \frac{\gamma - \dfrac{2\Delta}{a-2b+c}}{-2a-b}.$$

73. The lines $l\alpha + m\beta + n\gamma$, $l'\alpha + m'\beta + n'\gamma$ intersect in the point

$$\frac{\alpha}{mn' - m'n} = \frac{\beta}{nl' - n'l} = \frac{\gamma}{lm' - l'm} = \frac{2\Delta}{a(mn' - m'n) + \&c}.$$

The general equation to a line parallel to $l\alpha + m\beta + n\gamma$ is, when made homogeneous, $l\alpha + m\beta + n\gamma + \kappa(a\alpha + b\beta + c\gamma) = 0$.

The equation to the line joining the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ is

$$\alpha(\beta'\gamma'' - \beta''\gamma') + \beta(\gamma'\alpha'' - \gamma''\alpha') + \gamma(\alpha'\beta'' - \alpha''\beta') = 0.$$

This straight line may also be expressed by

$$\frac{\alpha - \alpha'}{\alpha'' - \alpha'} = \frac{\beta - \beta'}{\beta'' - \beta'} = \frac{\gamma - \gamma'}{\gamma'' - \gamma'}, \text{ or by } \frac{\alpha - \alpha''}{\alpha'' - \alpha'} = \frac{\beta - \beta''}{\beta'' - \beta'} = \frac{\gamma - \gamma''}{\gamma'' - \gamma'}.$$

The lines $l\alpha + m\beta + n\gamma$ (1), $l'\alpha + m'\beta + n'\gamma$ (2) are parallel if (2) can be put in the form $\lambda(l\alpha + m\beta + n\gamma) + \mu(a\alpha + b\beta + c\gamma)$. Eliminating λ and μ from the equations thus obtained, viz., $\lambda l + \mu a - l' = 0$ &c., we obtain the condition of parallelism. But another investigation will be given hereafter.

74. *To find the angle between the lines*

$$\frac{\beta}{m}=\frac{\gamma}{n}, \quad \frac{\beta}{m'}=\frac{\gamma}{n'}.$$

Make CA, CB axes of x and y; then the equations become (Art. 70)

$$\frac{x}{m}=\frac{y}{n}, \quad \frac{x}{m'}=\frac{y}{n'}.$$

Thus the angle is (Art. 40)

$$\tan^{-1}\frac{(mn' \sim m'n)\sin C}{mm'+nn'+(mn'+m'n)\cos C}.$$

The condition of perpendicularity is

$$mm'+nn'+(mn'+m'n)\cos C=0.$$

Thus the equation to the second line is, in the case of perpendicularity,

$$\frac{\beta}{n+m\cos C}+\frac{\gamma}{m+n\cos C}=0.$$

75. In the equation $\frac{x}{a}+\frac{y}{b}-1=0$ (1), let b become very great. Then the equation tends to the form $\frac{x}{a}-1=0$, and the line represented tends to become a parallel to Ox through the point $(a, 0)$. If however we make *both a and b* very great, the line moves off to a very great distance, and the equation (1) tends to the form $-1=0$ or $1=0$. It is convenient to imagine a line at infinity, or rather *the* line at infinity, corresponding to the equation

$$\text{constant}=0.$$

The direction of this line is wholly undetermined. It is only to be looked on as a line in which two other lines (*parallel* lines) may intersect. It intersects any line in one point only. Thus it meets the axis of x in the point $x=+\infty, y=0$, or, which is the same thing, the point $x=-\infty, y=0$.

Its trilinear equation, made homogeneous, is

$$\lambda(a\alpha+b\beta+c\gamma)=0, \text{ or simply } a\alpha+b\beta+c\gamma=0.$$

76. There are two considerations which may help the reader to receive what seems the paradox in Art. 75. The first is this: the equation $ax = ax + c$ in Algebra is formally the same as $c = 0$, but gives for x the value ∞. It may be looked upon as the limiting form of the equation

$$ax = arx + c \text{ when } r = 1.$$

The other is this: the asymptote of a hyperbola is a tangent. At which end is it a tangent? At both. Then the ends are the same, or we have a straight line meeting a conic section in more than two points.

77. We can now find the condition of parallelism of two lines $l\alpha + m\beta + n\gamma$, $l'\alpha + m'\beta + n'\gamma$ by a simple method. For if these lines be parallel their intersection lies on the line

$$a\alpha + b\beta + c\gamma.$$

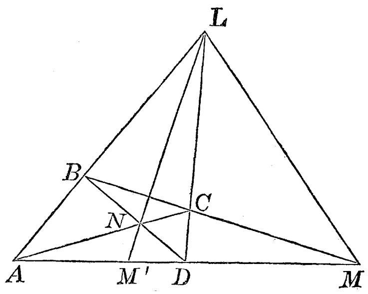
Thus the equations

$$l\alpha + m\beta + n\gamma = 0,\ l'\alpha + m'\beta + n'\gamma = 0,\ a\alpha + b\beta + c\gamma = 0$$

can coexist, for which the necessary and sufficient condition is

$$a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm) = 0.$$

78. Let A, B, C, D be any four points in a plane. We can draw three pairs of straight lines such that each pair shall include all the four points. Let these pairs be drawn, viz. AB, CD intersecting in L; AD, BC intersecting in M; and AC, BD intersecting in N. The quadrilateral $ABCD$ is now *complete*, and L, M, N are its three vertices.



Also LM, LM' form with LD, LA a harmonic pencil. To prove this it is sufficient to prove that if LN meet AM in M', then $AM'DM$ is a harmonic range. Now from the geometry of the triangle ALD we have

$$\frac{AM'}{DM'} \cdot \frac{DC}{LC} \cdot \frac{LB}{AB} = 1, \text{ and } \frac{AM}{DM} \cdot \frac{DC}{LC} \cdot \frac{LB}{AB} = 1,$$

therefore

$$\frac{AM'}{DM'} = \frac{AM}{DM},$$

or $AM'DM$ is a harmonic range.

Similarly ML, MB, MN, MD form a harmonic pencil, as also NB, NL, NC, NM.

Examples on Chapter V.

The equations to the sides BC, CA, AB of the triangle of reference are supposed to be $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and the lengths of the sides a, b, c.

1. Prove that the equation to the line joining A to the middle point of BC is $b\beta = c\gamma$.

2. Prove that the lines drawn from A, B, C to the middle points of BC, CA, AB, respectively, meet in a point; and find the coordinates of this point.

3. Find the equation to a line drawn through the point fgh parallel to the line $l\alpha + m\beta + n\gamma$. Ex. Let fgh be the point A.

4. Find the equation to the line joining the middle points of AB and AC and the equation to the parallel through A.

5. The equation to the line bisecting BC at right angles is

$$\beta \cos B - \gamma \cos C = \frac{a}{2} \sin (C - B).$$

6. Find the condition that the lines

$$l\alpha + m\beta + n\gamma, \; l'\alpha + m'\beta + n'\gamma, \; l''\alpha + m''\beta + n''\gamma$$

may meet in a point.

7. Prove that the lines bisecting BC, CA, AB at right angles meet in a point.

8. Find the condition that the points $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ may lie in a straight line.

9. Find the angle between the lines $l\alpha + m\beta + n\gamma$ and $l'\alpha + m'\beta + n'\gamma$. Deduce the condition of parallelism and the condition of perpendicularity.

10. Prove that the three bisectors of the angles of a triangle meet in a point, and find the equations to the line joining the feet of the bisectors.

11. Find the coordinates of a point which divides in a given ratio the line joining two other points.

12. Find the distance of the point fgh from the line

$$l\alpha + m\beta + n\gamma.$$

13. Indicate by a figure the points

$$\alpha = 2\beta = -3\gamma, \quad \alpha = -2\beta = 3\gamma,$$

and prove that if the sides of the triangle of reference be $3f$, $4f$, $6f$, these points lie on the same side of the line $\alpha + \beta = \gamma$. Also reduce the equation to the line to the form

$$\frac{\alpha - \alpha'}{c} = \frac{\beta - \beta'}{m} = \frac{\gamma - \gamma'}{n}.$$

14. In what ratio does the line $l\alpha = m\beta$ divide the line AB?

15. If BC, CA, AB be respectively parallel to $B'C'$, $C'A'$, $A'B'$, then AA', BB', CC' meet in a point.

16. The equations to three lines BC, CA, AB are

$$u = 0,\ v = 0,\ w = 0,$$

and the equation to a line meeting them in L, M, N is

$$lu + mv + nw = 0.$$

Find the equations to AL, BM, CN, and prove that

$$\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1.$$

17. Let O (see Example 16) be the point where BM meets CN. Then find the equations to AO, LO, and prove that there is in the figure a harmonic pencil at each of the points A, O, L.

18. Through A, B, C lines are drawn to the point $l\alpha = m\beta = n\gamma$, meeting the opposite sides in L, M, N. Prove that

$$\frac{BL \,.\, CM \,.\, AN}{CL \,.\, AM \,.\, BN} = 1.$$

19. Prove that if $l\alpha + m\beta + n\gamma = 0$ cross the triangle of reference, then l, m, n cannot all have the same sign. Supposing this line to meet AB, AC, not produced, in N, M, find the lengths of AM, AN. Find also the distance of MN from A.

20. Prove that the middle points of the lines AC, BD, LM in the figure of Art. 78 lie in a straight line; as also those of AD, BC, NL; and those of AB, CD, MN.

21. If $u = 0$, $v = 0$, $w = 0$, $lu + mv + nw = 0$ represent AB, BC, CD, DA in the figure of Art. 78, what straight lines will be

represented by

$$lu - 2mv + nw = 0,\ \ 2lu - mv + nw = 0,$$
$$lu - mv + 2nw = 0,\ \ mv + nw = 0,$$
$$nw + lu = 0,\ \ lu + mv = 0\,?$$

22. If there be two triangles ABC, $A'B'C'$, such that AA', BB', CC' meet in a point, then the intersections of corresponding sides will lie in a straight line, and conversely.

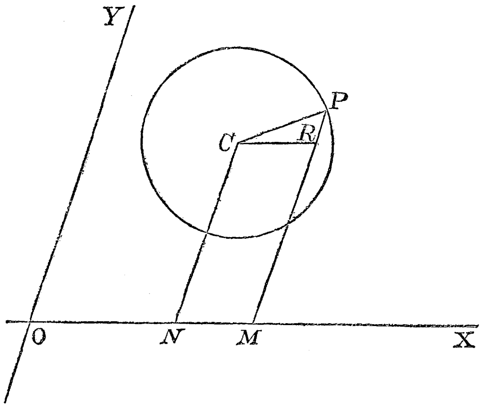
23. If L, M, N be taken in BC, CA, AB so that AL, BM, CN meet in a point; and if L', M', N' be the respective intersections of BC, MN; CA, NL; AB, LM; then L', M', N' will lie in a straight line.

CHAPTER VI.

THE CIRCLE.

79. We shall look upon the circle as a curve, calling that a circle which Euclid calls the circumference of a circle.

Let (a, b) be the centre C of a circle referred to axes OX, OY inclined at an angle ω, and let (x, y) be any point P on the circle. Also let the radius $= c$.



Draw CN, PM parallel to OY and CR parallel to OX. Then

$$CR^2 + CP^2 - 2\,CR \,.\, RP \cos CRP = CP^2,$$

or, $$(x-a)^2 + (y-b)^2 + 2\,(x-a)\,(y-b)\cos\omega = c^2 \ldots\ldots (1).$$

This is the relation existing between the x and y of any point on the curve, and is therefore the equation to the curve. (Art. 21.) Equation (1) may also be written

$$x^2 + y^2 + 2xy\cos\omega - 2\,(a + b\cos\omega)\,x - 2\,(a + b\cos\omega)\,y$$
$$+ a^2 + b^2 + 2ab\cos\omega - c^2 = 0 \ldots\ldots (2).$$

Thus the equation to a circle can always be reduced to the form

$$x^2 + y^2 + 2xy \cos \omega + Ax + By + C = 0 \ldots\ldots\ldots\ldots(3),$$

A, B, C being constants.

80. With rectangular axes the equation is

$$(x-a)^2 + (y-b)^2 = c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

or, $$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Thus the equation to a circle can always be reduced to the form

$$x^2 + y^2 + Ax + By + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If the origin be on the circumference, then (1) is satisfied by $x = 0$, $y = 0$.

Hence $a^2 + b^2 - c^2 = 0$, and (1) becomes

$$x^2 + y^2 = 2(ax + by)\, x \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If, besides, the axis of x be a tangent, $a = 0$, and the equation is

$$x^2 + y^2 = 2by \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly $$x^2 + y^2 = 2ax \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

represents a circle passing through the origin and touching the axis of y.

If the origin be centre the equation is

$x^2 + y^2 = c^2 \ldots(7)$, (with oblique axes $x^2 + y^2 + 2xy \cos \omega = c^2$).

81. With rectangular axes every equation of the form

$x^2 + y^2 + Ax + By + C = 0 \ldots(1)$ represents a circle.

For (1) may be written

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2 + B^2}{4} - C \ldots\ldots\ldots\ldots(2).$$

And this represents a circle with centre $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ and radius $\sqrt{\frac{A^2+B^2}{4} - C}$.

We suppose A and B real. Thus the circle has a real centre.

The radius is real if $\frac{A^2+B^2}{4} - C$ be positive. The radius, and therefore the circle, is imaginary if $\frac{A^2+B^2}{4} - C$ be negative.

The real and imaginary classes of circles are separated by the case in which $\frac{A^2+B^2}{4} - C$ is zero, that is, by a circle with centre $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ and radius indefinitely small, or, the 'point-circle' $\left(-\frac{A}{2}, -\frac{B}{2}\right)$.

82. This point-circle is *also* the two imaginary straight lines

$$x+\frac{A}{2}+\sqrt{-1}\left(y+\frac{B}{2}\right)=0, \qquad x+\frac{A}{2}-\sqrt{-1}\left(y+\frac{B}{2}\right)=0.$$

So $x^2+y^2=0$ is both the point-circle $(0, 0)$ and the two imaginary straight lines $x+y\sqrt{-1}=0$, $x-y\sqrt{-1}=0$.

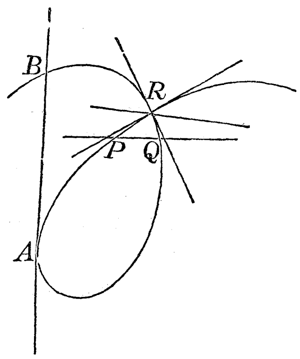
Tangents and Normals.

83. A tangent to a curve may be thus defined:

Let two points be taken on a curve, and a secant (or chord) drawn through them; let one point remain fixed and let the other move up to it along the curve. Then the limiting position of the secant is the tangent to the curve at the fixed point.

According to Euclid a straight line touches a circle when it meets the circle in one point only. As far as the circle is concerned, our definition agrees with his; but if we are to speak of curves in general, Euclid's definition must be enlarged.

Take a curve with a loop, for instance. A line AB may touch the loop and cut the curve elsewhere.



It is also necessary that one of the points spoken of in our definition be *fixed*. It would not do to say 'let the two points move along the curve up to some third point; then the limiting position of the chord will be the tangent to the curve at the third point.' For if R be a double point and RP, RQ the two curve lines which cross at R, the chord PQ, when P and Q have moved up to R and there coincided, is not necessarily a tangent: indeed its direction is not determined. But if the chords RP, RQ be drawn and PQ move up to R, RP, RQ are ultimately tangents at R.

The *normal* at any point of a curve is a straight line drawn through the point perpendicular to the tangent to the curve at the point.

A tangent to a curve at any point may be called the *direction* of the curve at the point, and thus the normal at the point is a perpendicular to the direction of the curve at the point.

84. *To find the equations to the tangent and normal at any point* x, y *of the circle* $x^2 + y^2 = c^2$.

Let x_2y_2 be another point on the curve. The equation to the chord joining x_1y_1 and x_2y_2 is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \quad\text{.............................(1).}$$

Now $x_1^2+y_1^2=c^2$, and $x_2^2+y_2^2=c^2$, because the two points are on the curve, so that the coordinates of each of them satisfy the equation to the curve. Therefore

$$x_2^2-x_1^2+y_2^2-y_1^2=0.$$

Therefore
$$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2}{y_1+y_2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus (1) becomes

$$\frac{y-y_1}{x_1+x_2}+\frac{x-x_1}{y_1+y_2}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let x_2y_2 move up to x_1y_1. Then ultimately $x_2=x_1$ and $y_2=y_1$.

Thus (3) becomes

$$\frac{y-y_1}{x_1}+\frac{x-x_1}{y_1}=0,$$

or
$$yy_1+xx_1=x_1^2+y_1^2,$$

or
$$xx_1+yy_1=c^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This is the equation to the tangent at x_1y_1.

The equation to a perpendicular line must be of the form

$$\frac{x}{x_1}-\frac{y}{y_1}=\text{a constant (Art. 45).}$$

The normal is that perpendicular which passes through x_1y_1; and therefore its equation is

$$\frac{x}{x_1}-\frac{y}{y_1}=\frac{x_1}{x_1}-\frac{y_1}{y_1},$$

or
$$\frac{x}{x_1}=\frac{y}{y_1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Of course equation (5) could have been inferred at once from what Euclid has taught us about the circle, and (4) could have been deduced from (5) just as we have deduced (5) from (4). But we have preferred to illustrate our definition of a tangent.

85. *From any external point there can be drawn two tangents to a circle.*

Let the centre of the circle be made the origin, and let c be the radius. The equation is

$$x^2 + y^2 = c^2.$$

Let hk be the external point, and $x'y'$ a point on the circle such that the tangent there drawn passes through hk.

The equation to the tangent is

$$xx' + yy' = k^2,$$

and since this is satisfied by h, k,

$$x'h + y'k = c^2 \text{ ....} (1);$$

also, since $x'y'$ is on the circle,

$$x'^2 + y'^2 = c^2 \text{ ....} (2).$$

From (1) and (2) we can find x' and y'. The equation for x' is

$$x'^2(h^2 + k^2) - 2c^2hx' + c^2(c^2 - k) = 0,$$

of which both roots are real, as the reader will find after a short algebraical process, since $h^2 + k^2 > c^2$.

To each value of x' corresponds a value of y', determined from (1). Thus there are two points of contact such that the tangents there drawn pass through the given external point.

86. The line joining the two points is called the *chord of contact.*

Suppose we want the equation to the chord of contact corresponding to the point (h, k). Let x_1y_1, x_2y_2 be the two points of contact. Then $x_1h + y_1k = c^2$ (1), and $x_2h + y_2k = c^2$ (2), because the tangents at x_1y_1 and x_2y_2 pass through hk. Therefore x_1y_1 and x_2y_2 each lie on the straight line $xh + yk = c^2$ (3). (Equations (1) and (2) only assert this fact in algebraic language.) Therefore (3) is the equation to the chord of contact.

87. When hk is on the circle, the equation $xh+yk=c^2$ represents the tangent at hk. The chord of contact belonging to hk has, in fact, become this tangent. But when hk has passed within the circle the tangents are no longer real; nevertheless the line $xh+yk=c^2$ is real, that is, the chord of contact is real and can be drawn. In Art. 60, we saw that imaginary lines may contain real points. It is also true that real lines may contain imaginary points. Thus the line $2x=3y$ contains the point $(3\sqrt{-1}, 2\sqrt{-1})$, and, in short, an infinite number of imaginary points. When hk is within the circle the chord of contact passes through two imaginary points of contact, the points where the chord meets the circle.

88. But we can assign another meaning to the equation $xh+yk=c^2$ which will not involve mystery.

Through h, k draw any *real* chord and draw tangents at its extremities. Let these tangents meet in x_1y_1, then the equation to the chord, by Art. 86, is $xx_1+yy_1=c^2$. But this line passes through hk, and therefore $x_1h+y_1k=c^2$. This last equation states that x_1y_1 lies on the straight line $xh+yk=c^2$. Hence, if through any fixed point hk chords be drawn to a circle $(x^2+y^2=c^2)$ and tangents be drawn at the extremities of the chord, the locus of their intersection is a straight line $(xh+yk=c^2)$.

89. Conversely, if from any point $(x'y')$ in a straight line $(Ax+By+C=0)$ tangents be drawn to a circle $(x^2+y^2=c^2)$, the chord of contact will pass through a fixed point.

For the chord of contact for $x'y'$ is $xx'+yy'=c^2$, and since $Ax'+By'+C=0$, this may be written

$$x'\left(x-\frac{Ay}{B}\right)-\left(\frac{yC}{B}+c^2\right)=0,$$

a form which proves that the chord of contact passes through the intersection of the fixed lines

$$x-\frac{Ay}{B}=0, \quad \frac{yC}{B}+c^2=0.$$

Thus the co-ordinates of the fixed point are

$$x=-\frac{Ac^2}{C}, \quad y=-\frac{Bc^2}{C}.$$

90. Any equation $Ax + By + C = 0$ can be written in the form

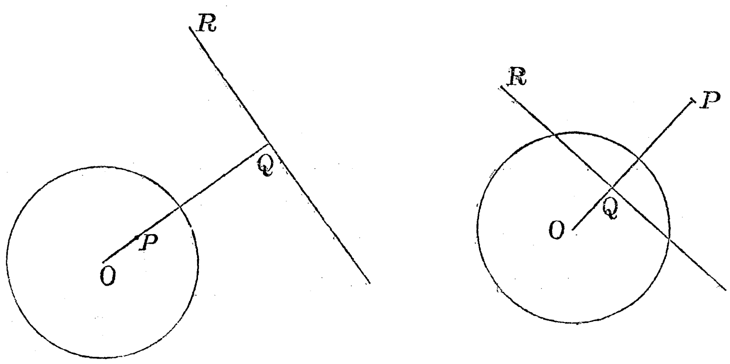
$$xh + yk = c^2,$$

the h and k being $-\frac{Ac^2}{C}$, $-\frac{Bc^2}{C}$. Thus to every point hk there corresponds a line $xh + yk = c^2$, and to every line $Ax + By + C$ there corresponds a point $-\frac{Ac^2}{C}$, $-\frac{Bc^2}{C}$.

The line is the locus of the intersection of tangents at the ends of any chord through the point, and, when the point is outside the circle, the line is also a real chord of contact corresponding to the point.

The point is called the *Pole* of the line, and the line the *Polar* of the point. We do not here define the terms *Pole* and *Polar*, but introduce them for convenience.

91. Let the point h, k be in the axis of x; then $k = 0$, and the equation to the polar is $xh = c^2$. Thus the intercept on the axis of y is $\frac{c^2}{h}$. We now have a convenient geometrical construction for drawing the polar of a given point.



Let O be the centre of the circle, P the given point. Join OP, and in OP, produced if necessary, take Q such that $OP \cdot OQ$

$= (\text{radius})^2$. Through Q draw a line QR at right angles to OP. QR is the polar of P.

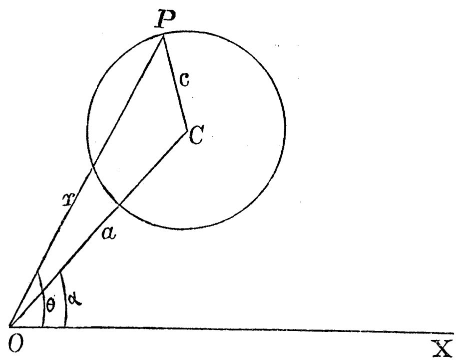
In like manner, if we wish to find the pole of a line QR, we draw OQ at right angles to QR, and in OQ, produced if necessary, take P so that $OQ \,.\, OP = (\text{radius})^2$; then P is the pole of QR.

If PS be perpendicular to OP, PS is the polar of Q. We thus see that if one point P lie on the polar of Q, then Q lies on the polar of P.

92. This also appears from the equations. For let $x^2+y^2=c^2$ be the circle, hk the point P, and $h'k'$ the point Q. The polars of P and Q are $xh+yk=c^2$, $xh'+yk'=c^2$. If P be on the polar of Q, $hh'+kk'=c^2$, which equation states that Q lies on the polar of P.

Polar Co-ordinates.

93. *To find the polar equation to the circle.*



Let a, α be the polar co-ordinates of the centre C, r, θ those of any point P on the curve, c the radius; then

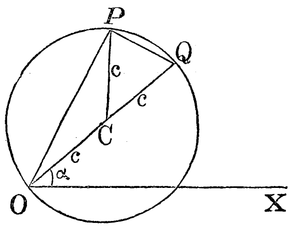
$$CP^2 = OC^2 + OP^2 - 2\,OC\,.\,OP \cos POC,$$

i.e. $$c^2 = a^2 + r^2 - 2ar\cos(\theta - \alpha) \qquad (1).$$

This is the polar equation to the circle.

If OX pass through C, $\alpha = 0$, and the equation is

$$c^2 = a^2 + r^2 - 2ar\cos\theta \qquad (2).$$



If O be on the circumference, then $a = c$, and (1) becomes

$$r = 2c \cos(\theta - \alpha) \quad\text{.....................}\quad (3),$$

which is geometrically deducible from the right-angled triangle POQ.

If O be on the circumference and OX a diameter, then $\alpha = 0$, and (3) becomes

$$r = 2c \cos\theta \quad\text{.......................}\quad (4).$$

If O be on the circle and OX a tangent, $\alpha = \frac{\pi}{2}$, and (3) becomes

$$r = 2c \sin\theta \quad\text{..........................}\quad (5).$$

If O be the centre, $a = 0$, and (1) becomes

$$r^2 = c^2 \text{ or } r = c \quad\text{........................}\quad (6).$$

Equations (4), (5), (6) can of course be obtained independently, by geometry.

The polar equation to the circle can always be put in the form

$$r^2 + Ar\cos\theta + Br\sin\theta + C = 0.$$

94. As an example of Polar Co-ordinates applied to the circle, let us take Euclid, III. 35, 36.

Corresponding to any value of θ in equation (1) of the last article there are two values of r, whose product is $a^2 - c^2$, a constant.

If O be without the circle the product of the two values of r is positive. If O be within the circle the product is negative, since $a^2 < c^2$, and in this case the values of r have opposite signs.

EXAMPLES ON CHAPTER VI.

1. Find the centre and radius of the circle
$$x^2 + y^2 + 15x + 30 = 0.$$

2. Find the equation to the circle described on the line joining x_1y_1 and x_2y_2 as diameter.

3. Prove that the equation
$$x^2 + y^2 + 2xy \cos \omega + Ax + By + C = 0,$$
if the axes be inclined at an angle ω, represents a circle.

4. What must be the inclination of the axes in order that
$$x^2 + y^2 - 2xy \cos \alpha + Ax + By + C = 0$$
may represent a circle?

5. What in order that
$$x^2 + y^2 + xy = hx + ky$$
may represent a circle? Determine the centre and radius.

6. Find the intercepts which the circle in (3) makes on the co-ordinate axes, and shew that if the circle touches the axis of x, $A^2 = 4C$.

7. If from, or through, any point O chords AB, CD be drawn to a circle, the rectangle $OA \,.\, OB =$ the rectangle $OC \,.\, OD \ldots (\omega)$.

8. Find the relation between A, B, and C in order that the circle in (3) may touch both axes; and prove that two tangents drawn to a circle from any external point are of equal length.

9. The equation to the tangent to the curve
$$(x - a)^2 + (y - b)^2 = c^2$$
at the point $x'y'$ is
$$(x - a)(x' - a) + (y - b)(y' - b) = c^2.$$
Find also the equation to the normal.

10. Prove that Euclid, III. 32, is a particular case of an earlier and more general proposition in the same Book.

11. If α, α' be vectorial angles of two points on the circle $x^2+y^2=c^2$, the equation to the chord joining the points is

$$x \cos \frac{\alpha+\alpha'}{2} + y \sin \frac{\alpha+\alpha'}{2} = c \cos \frac{\alpha \sim \alpha'}{2}.$$

12. Hence find the equation to the tangent at the point α.

13. Apply Art. 46 to find the condition that the line

$$Ax+By+C$$

may touch the circle

$$(x-a)^2+(y-b)^2=c^2.$$

14. Does the line $2x+3y=0$ touch the circle

$$x^2+y^2=2x+3y\,?$$

15. Prove that two tangents can be drawn to a circle in a given direction, and that the tangents to $x^2+y^2=c^2$ in direction $y=mx$ are

$$y=mx \pm c\sqrt{1+m^2}.$$

16. The angle in a semicircle is a right angle.

17. From an external point tangents are drawn to the circle $x^2+y^2=2ax$. Find the equation to the chord of contact.

18. Find the equation to the tangent to the circle

$$x^2+y^2+2xy\cos\omega=c^2$$

at the point $x'y'$.

19. A point moves so that the sum of the squares of its distances from the sides of a square is constant. Find the locus of this point. Shew that the *position* of the locus does not depend on the magnitude of the constant sum.

20. A and B are given points, and $AP : BP$ is a constant ratio. Prove that the locus of P is a circle.

Also, given that the locus of P is a circle, find geometrically the circle's position and magnitude.

21. A point moves so that the sum of the squares of its distances from any number of given points is constant. Prove that the locus of this point is a circle.

22. Find the centre of this circle in the case of *two* given points. Prove that, if there be *four* given points A, B, C, D, the centre is at the middle point of the line joining the middle points of the diagonals of the quadrilateral $ABCD$.

23. Find the polar of the point $(1, -2)$ with respect to the circle $x^2+y^2=9$, and the pole of the straight line $x+y=1$ with respect to the same circle.

24. Apply Art. 91 to shew that the polar of the point $x'y'$ with respect to the circle $(x-a)^2+(y-b)^2=c^2$ is

$$(x-a)(x'-a)+(y-b)(y'-b)=c^2.$$

25. Find the polar equation to the tangent to the circle $r=a$ at the point α.

26. Through a fixed point chords are drawn to a circle. Prove that the locus of their middle points is a circle, and find the position and magnitude of this circle.

27. Through a fixed point O is drawn a chord OPQ to a circle, and in PQ is taken R, so that OP, OR, OQ are in Harmonic Progression. Shew that the locus of R is a straight line.

28. Also, assuming that the locus of R is a straight line, prove that if O be without the circle the locus is the chord of contact corresponding to O.

29. In the radius vector OP of a circle P' is taken so that $OP' : OP$ is a constant ratio. Find the locus of P', and prove by Art. 83 that the tangents to the given circle from O are also tangents to this locus.

30. If P' be so taken that the rectangle $OP \cdot OP'$ is constant, what will be the locus of P'? Examine the case in which O lies on the given circle.

31. In OP the radius vector of a straight line is taken P' so that $OP \cdot OP'$ is constant. Find the locus of P'.

32. Shew that the equation to a circle referred to two tangents of length c, and inclined at an angle ω, is

$$(x^2 + y - c)^2 = 4xy \sin^2 \frac{\omega}{2}. \quad \text{(See Ex. 8.)}$$

33. Apply Ex. 32 to prove that if TA, TB be two tangents to a circle, and P any point on the circle, the perpendicular from P on AB is a geometric mean between the perpendiculars from P on TA, TB.

CHAPTER VII.

THE CIRCLE (*continued*).

95. LET S denote $x^2+y^2+Ax+By+C$; then $S=0$ is the equation to a circle whose centre is $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ and radius $\sqrt{\frac{A^2+B^2}{4}-C}$. Call the co-ordinates a, b, and the radius c; then S denotes $(x-a)^2+(y-b)^2-c^2$.

So long as the S of a point is positive the point is without the circle; when the S of the point vanishes the point is on the circle; when the S of a point is negative, the point is within the circle. $S=$ a constant is the equation to a concentric circle. The S of any point (as is evident from Euclid, I. 47) is the square of the tangent from that point to the circle, or the rectangle under the segments of any chord through that point. The tangent is impossible and the rectangle is negative when the point is within the circle.

96. Let S' denote

$$x^2+y^2+A'x+B'y \text{ or } (x-a')^2+(y-b')^2-c'^2,$$

then $S-S'=0$ is a linear equation; also any values of x and y which make both S and S' vanish make $S-S'$ vanish. Thus $S-S'=0$ is the equation to the common chord of the circles S, S'. The unabridged form is

$$(a'-a)\,x+(b'-b)\,y+a'^2-a^2+b'^2-b^2=c'^2-c^2.$$

Thus the common chord is perpendicular to the line

$$\frac{x-a}{a'-a}=\frac{y-b}{b'-b},$$

that is, to the line joining the centres.

97. The two circles may not intersect, the reader may say. Still their imaginary points of intersection lie on a real common chord. In fact, if a line and circle be drawn in one plane, the line always intersects the circle in two points, real, coincident, or imaginary. Two curves of the m^{th} and n^{th} degrees, drawn in the same plane, intersect in mn points, of which, if mn be even, all *may* be imaginary. Two circles intersect in 4 points, of which two may be real.

98. The *radical axis* of two circles is their common chord. It is the locus of points from which the tangents to the circles are equal, and (as the reader can prove by a figure) divides the line of centres so that the difference of the squares of the segments equals the difference of the squares of the radii.

99. The equation $S-\lambda S'=0$ represents a circle passing through the intersection of S and S': by giving a proper value to λ, we can thus represent *any* circle passing through those intersections. For let S_1, S_1' be values of S and S' for any point $x_1\,y_1$, then if the circle $S-\lambda S'$ passes through $x_1 y_1$, $\lambda=\dfrac{S_1}{S_1'}$. Thus $S-\dfrac{S_1}{S_1'}\,.\,S'=0$ is a circle passing through the common points of S and S' and an arbitrary point $x_1 y_1$. And by varying $x_1 y_1$ we can represent *any* circle passing through the other two points.

Tangents.

100. We shall now give another method of finding the equation to a tangent to a circle at a given point.

Let x_1y_1 be the point, and

$$x^2+y^2+2xy\cos\omega+Ax+By+C=0 \;\ldots\ldots\ldots(1)$$

the circle.

Transfer the origin to x_1y_1; then the new equation is

$$(x+x_1)^2+(y+y_1)^2+2(x+x_1)(y+y_1)\cos\omega$$
$$+A(x+x_1)+B(y+y_1)+C=0;$$

or, since x_1y_1 is on the curve,

$$x^2+y^2+2xy\cos\omega+2\left(x_1+y_1\cos\omega+\frac{A}{2}\right)x$$
$$+2\left(y_1+x_1\cos\omega+\frac{B}{2}\right)y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In polar co-ordinates this is

$$r^2(l^2+m^2+2lm\cos\omega)+2r\left\{l\left(x_1+y_1\cos\omega+\frac{A}{2}\right)\right.$$
$$\left.+m\left(y_1+x_1\cos\omega+\frac{B}{2}\right)\right\}=0 \;\ldots\ldots\ldots(3).$$

If we wish to know the length of a radius vector drawn in any given direction, we must substitute the corresponding given values of l, m in (3). We have then a quadratic in r, one root of which is always zero whatever l and m may be. But the other value will also be zero if we take l and m such that

$$2l\left(x_1+y_1\cos\omega+\frac{A}{2}\right)+2m\left(y_1+x_1\cos\omega+\frac{B}{2}\right)=0 \;\ldots\ldots(4).$$

Thus (4) is the polar equation to the tangent at the origin. The corresponding equation in Cartesians is

$$2x\left(x_1+y_1\cos\omega+\frac{A}{2}\right)+2y\left(y_1+x_1\cos\omega+\frac{B}{2}\right)=0 \;\ldots(5),$$

(*which equates to zero the terms of lowest dimensions* in (3)).

We return to the old origin by writing $x-x_1$, $y-y_1$ for x and y in (5).

The result is

$$2x\left(x_1 + y_1 \cos\omega + \frac{A}{2}\right) + 2y\left(y_1 + x_1 \cos\omega + \frac{B}{2}\right)$$

$$= 2\left(x_1^2 + y_1^2 + 2x_1y_1 \cos\omega + \frac{A}{2}x_1 + \frac{B}{2}y_1\right)$$

$$= 2\,(x_1^2 + y_1^2 + 2x_1y_1 \cos\omega + Ax_1 + By_1 + C)$$

$$- (Ax_1 + By_1 + 2C) = -(Ax_1 + By_1 + 2C),$$

or

$$2\,(xx_1 + yy_1) + 2\cos\omega\,(xy_1 + yx_1)$$
$$+ A\,(x + x_1) + B\,(y + y_1) = 0 \quad \ldots\ldots\ldots\ldots\ldots(6).$$

If we change x_1, y_1 to x, y, and x, y to x_1, y_1, this equation is unaltered. Of course this symmetry exists also in the less general cases.

The clause in italics points to the following principle:

Whenever the equation to a curve is rational and integral, and the curve passes through the origin, the equation to the tangent lines at the origin is found by equating to zero the terms of lowest dimensions in the equation to the curve.

Ex. $x^2 + y^2 = 2cy$. Here $y = 0$ is the tangent line at the origin.

Suppose we want to find the equation to the tangent to the circle $x^2 + y^2 = 2cy$ at the point x_1y_1. Transfer the origin, and the equation becomes

$$x^2 + y^2 + 2xx_1 + 2yy_1 = 2cy.$$

The tangent at the new origin is $xx_1 + yy_1 = cy$.

Transfer back, and the equation required is

$$(x - x_1)\,x_1 + (y - y_1)\,y_1 = c\,(y - y_1),$$

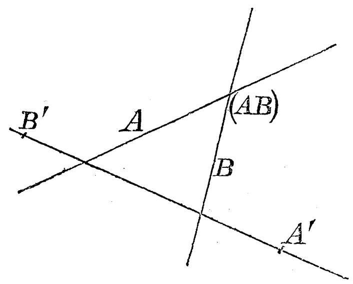
or $$xx_1 + yy_1 = c\,(y + y_1), \text{ or } x_1^2 + y_1^2 - cy_1 = cy_1.$$

If tangents be drawn to the circle (1) from an external point x_1y_1, then (6) will be the equation to the chord of contact. (We leave the proof to the reader.) Thus (6) is the equation to the *polar* of x_1y_1 with regard to the circle (1).

Poles and Polars.

101. *The intersection of two lines is the pole of the line which joins the poles of the lines.*

Let A, B be the two lines, and let A', B' be their poles. Let (AB) denote the intersection of A and B.



Then the line A passes through the point (AB); therefore the point A' lies on the polar of (AB). Similarly the point B' lies on the polar of (AB); therefore $A'B'$ *is* the polar of AB.

If a line A pass through a fixed point P', its pole A' lies on a fixed line P; and if a point A' lie on a fixed line P, its polar passes through a fixed point P'. If any number of lines meet in a point their poles are in a straight line, and *vice versâ*.

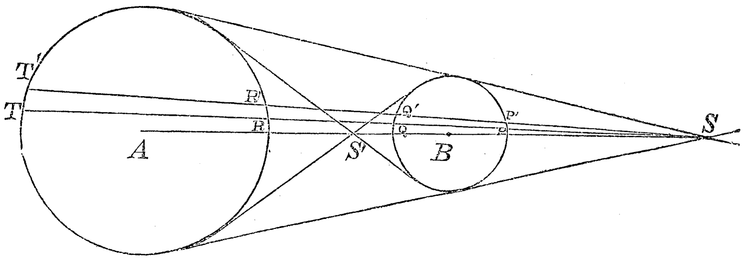
The equation $OQ \,.\, OP = c^2$ in Art. 91, shews that as P moves towards O, the polar of P recedes from O. When $OP = 0$, $OQ = \infty$. Thus *the centre of the circle is the pole of the line at infinity.*

The reader will easily prove, by taking the simple form $x^2 + y^2 = c^2$, that the perpendicular from a point P on the polar

of a point Q is to the perpendicular from Q on the polar of P as the perpendicular from the centre O on the polar of Q is to the perpendicular from the centre O on the polar of P. Let $P(Q)$ denote the first of these lines; then we may thus state the proposition:

$$\frac{P(Q)}{Q(P)} = \frac{C(Q)}{C(P)}.$$

102. *Centres of Similitude.*



Let A, B be centres of two circles of which the radii are a, b, a being greater than b. Divide AB internally and externally at S', S in the ratio $a:b$; then S', S are the internal and external *Centres of Similitude* of the two circles, and the two external common tangents meet in S, and the two internal common tangents (which may be imaginary) in S'. Also if $SPQRT$ be any line drawn through S cutting the circles, the radii AR, BP are parallel, as also the radii AT, BQ. Thus $SP:SR = SQ:ST = b:a$, a constant ratio for *all* such lines through S. Thus the circle A could be obtained from the circle B by producing each radius vector SP (S being considered a pole) of the circle B to a point R such that $SR:SP = a:b$. Had there been any curve whatsoever in the stead of B such a construction would have given a similar curve, similarly placed (and magnified in linear dimensions in the ratio $a:b$). The term 'centre of similitude' is thus illustrated.

Again, the rectangles $SQ.SR$, $SP.ST$ are equal and constant, being to the rectangle $SP.SQ$ as $a:b$. Thus, if $SP'Q'R'T'$ be another line through S, the points Q, Q', R', R lie in a circle,

and so do the points P, P', T', T. If QQ', RR' meet in Z, $ZQ \,.\, ZQ' = ZR \,.\, ZR'$ (Euclid, III. 36): *therefore Z lies on the radical axis of the circles A, B.* Similarly TT', PP' intersect on the radical axis.

Let there be a third circle with centre C drawn. There are now six centres of similitude, and it readily follows from the geometry of the triangle ABC that these centres of similitude lie three and three in four straight lines. These lines are called *axes of similitude.*

Examples on Chapter VII.

1. Is the origin within or without the circle

$$2(x^2+y^2)+3x-5y+7=0\,?$$

2. Prove, by any means, that the circles

$$(x-a)^2+(y-b)^2=c^2, \quad (x-a')^2+(y-b')^2=c'^2,$$

touch one another if

$$(a-a')^2+(b-b')^2=(c\pm c')^2.$$

3. Find the condition that the circles

$$\alpha(x^2+y^2)+Ax+By+C, \quad \alpha'(x^2+y^2)+A'x+B'y+C',$$

may touch one another.

4. ABC is a triangle, and AB, AC are taken for axes. Find the equation to the circumscribing circle, and the equations to the tangents at A, B, C. Prove that the tangent at an angular point A makes angles B, C with AC, AB.

5. A circle circumscribes the triangle whose vertices are x_1y_1, x_2y_2, x_3y_3. Find its equation.

6. Find the equation to the polar of the origin with regard to the circle $x^2+y^2+Ax+By+C$.

7. Find the equations to tangents to the following curve at the point $x'y'$:

(1) $ay=x^2$. (2) $a^2y=x^3$.

(3) $\dfrac{y}{a}=\sin\dfrac{x}{b}$. (4) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

8. ABC, abc are two triangles such that A, B, C are poles of bc, ca, ab. Prove that a, b, c are poles of BC, CA, AB.

Such triangles are called *copolar* triangles.

9. If ABC, abc be copolar triangles, then Aa, Bb, Cc meet in a point, and the intersections of BC, bc; CA, ca; AB, ab lie in one straight line.

10. Hence prove that the lines joining the angular points of a triangle to the points of contact of the inscribed circle meet in a point.

11. What are the conditions necessary in order that the general equation

$$ax^2+by^2+c+2a'y+2b'x+2c'xy=0,$$

may represent a circle? (ω).

12. If four fixed points be taken on a circle and joined to any fifth on the circle, the pencil formed will have a constant anharmonic ratio. (Euclid, III. 21.)

13. Find the ratios in which the line joining the points xy, $x'y'$ is cut by the circle $x^2+y^2=c^2$, and hence find the equation to the tangent at a point on the circle and the equation to two tangents from an external point.

14. In the fig. of Art. 77, let A, B, C, D lie in a circle, and let LA, LD be axes; let $LA = a$, $LB = b$, $LC = c$, $LD = d$, and the angle $ALD = \omega$. Then prove that the equation to the circle is

$$\frac{x^2}{aa'} + \frac{y^2}{bb'} + 2xy \cos \omega - x\left(\frac{1}{a} + \frac{1}{a'}\right) - y\left(\frac{1}{b} + \frac{1}{b'}\right) + 1 = 0,$$

and prove that of the points L, N, D each is the polar of the line joining the other two. (Ex. 29, Chap. VI.)

CHAPTER VIII.

THE PARABOLA.

103. A *conic section* is the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. The conic section is a *parabola*, an *ellipse*, or a *hyperbola*, according as this constant ratio is equal to, less than, or greater than unity. The fixed point is called the *focus*, the fixed line the *directrix*, and the constant ratio the *eccentricity*.

The propriety of the term 'conic section' rests on this, that if a right circular *cone* be cut by a plane, the *section* will be one of these aforesaid curves.

A curve whose equation is of the n^{th} degree is called a curve of the n^{th} degree.

All conic sections are of the second degree. For let xy be any point on the conic, and let fg be the focus, and $Ax + By + C = 0$ the directrix; also let $e : 1$ be the given ratio or eccentricity. Then xy's distance from fg is e times xy's distance from

$$Ax + By + C: \text{ that is,}$$

$$\sqrt{(x-f)^2 + (y-g)^2} = e \cdot \frac{Ax + By + C}{\sqrt{A^2 + B^2}},$$

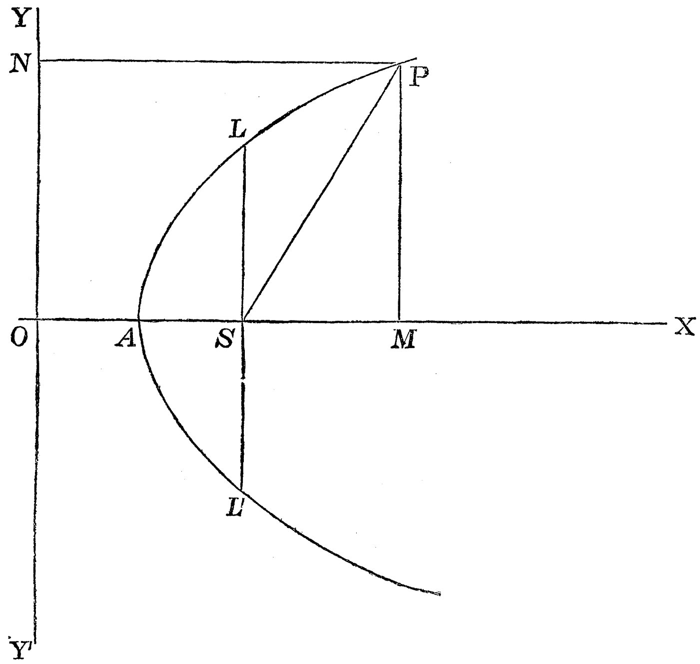
or
$$(x-f)^2 + (y-g)^2 = e^2 \cdot \frac{(Ax + By + C)^2}{A^2 + B^2}.$$

This is the equation to the conic section, and it is rational, integral, and of the second degree. With oblique axes the equation would still be rational, integral, and of the second degree. Thus all conic sections are curves of the second degree.

It will be proved hereafter that all curves of the second degree are conic sections.

104. To find the equation to the parabola in a simple form we may proceed thus.

Let S be the focus, and SO the perpendicular from S on the directrix YY'. Take OS, OY for axes of x and y. Let $OS=2a$,



and let x, y be co-ordinates PN, PM of any point P on the curve. Then

$$PS = PN; \ \therefore \ PS^2 = PN^2,$$

or,
$$PM^2 + SM^2 = OM^2,$$

or,
$$y^2 + (x-2a)^2 = x^2,$$

or,
$$y^2 = 4a\,(x-a) \qquad\qquad (1).$$

This is the equation to the curve.

Put $y=0$ in (1); then $x=a$, that is, if A be the point where the curve meets the axis of x, $OA = AS = a$.

The point A is called the *vertex* of the curve and the line Ax the *axis* of the curve.

If we transfer the origin to the point A without altering the direction of the axes, equation (1) becomes

$$y^2 = 4ax \quad\text{..........................}\quad (2),$$

and this is the form we shall chiefly use.

105. The curve $y^2 = 4ax$ passes through the origin and the line $x = 0$ is the tangent at the origin. (Art. 100.)

For every positive value of x there are two values of y, equal in magnitude and of opposite sign: thus the curve is symmetrical with respect to the axis of x. Also if x be negative, y is impossible. Thus the curve lies wholly on the Ox side of the axis of y.

If x be infinitely great, so is y. Thus the curve is of infinite size.

106. *To find the equation to the tangent at the point* $x'y'$.

The equation to the chord joining $x'y'$ and $x''y''$ is,—since

$$y'^2 - 4ax' = 0 = y''^2 - 4ax'',$$

so that

$$\frac{x'' - x'}{y'' + y'} = \frac{y'' - y'}{4a},\text{—}$$

$$\frac{x - x'}{y'' + y'} = \frac{y - y'}{4a}.$$

When $x''y''$ moves up to, and coincides with, $x'y'$, the chord becomes the tangent at $x'y'$, and its equation, since ultimately $y'' = y'$, is

$$\frac{x - x'}{y'} = \frac{y - y'}{2a}, \text{ or } yy' = 2ax + y'^2 - 2ax',$$

or

$$yy' = 2a(x + x') \quad\text{........................}\quad (1),$$

since

$$y'^2 = 4ax'.$$

Of course this equation can be obtained by the method of Art. 100.

The normal at $x'y'$ is perpendicular to the line

$$yy' = 2a(x + x').$$

Its equation is therefore of the form $2ay + y'x =$ a constant.

But the normal also passes through $x'y'$; therefore the constant is $2ay' + y'x'$.

Therefore the equation to the normal at $x'y'$ is

$$2ay + y'x = y'(2a + x') \dots\dots\dots\dots\dots\dots (2).$$

107. The equation $yy' = 2a(x + x')$, since $x' = \frac{y'^2}{4a}$, may be written

$$y = \frac{2a}{y'}x + \frac{y'}{2a}, \text{ or, if } \frac{2a}{y'} \text{ be denoted by } m,$$

$$y = mx + \frac{a}{m} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Here m is the tangent of the angle which the line makes with the axis of x.

Conversely any line whose equation is of the form

$$y = x\tan\theta + a\cot\theta,$$

touches the parabola $y^2 = 4ax$, viz. at the point $y = 2a\cot\theta$,

$$x = a\cot^2\theta.$$

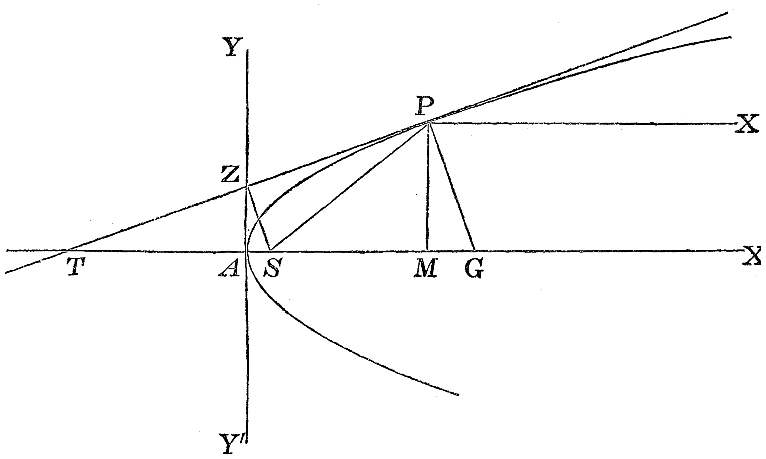
Equation (2) of the preceding article can likewise be expressed in terms of the inclination of the normal to the axis of x. For let that inclination be $\tan^{-1} m$, or, let $m = -\frac{y'}{2a}$, then $x' = \frac{y'^2}{4a} = am^2$, and the equation becomes

$$y = mx - 2am - am^3 \dots\dots\dots\dots\dots\dots (2).$$

108. Let PT, PG be tangent and normal at the point $P(x'y')$, and let PM be perpendicular to the axis of the curve.

In the equation $yy' = 2a(x + x')$, i.e. in the equation to PT, put $y = 0$, then

$$x = -x'. \quad \text{Thus } AT = AM.$$



Again, $SP = AM + AS$ (by definition) $= x' + a$;

$$\therefore \text{ since } AM + AS = AT + AS, \; SP = ST.$$

Thus the triangle STP is isosceles, and if PX' be parallel to the axis of the curve, SP, PX' are equally inclined to the tangent at P.

Again, in the equation to PG, or

$$2ay + y'x = y'(2a + x'), \text{ put } y = 0;$$

$$\text{then } x = 2a + x'. \quad \text{Thus } MG = 2a.$$

The double ordinate through the focus of a conic section is called the *latus rectum*. In $y^2 = 4ax$ if we put $x = a$ we get $y = \pm 2a$. Thus the latus rectum of this curve is in length $4a$, and MG is half the latus rectum.

Also $\qquad SG = 2a + x' - SA = a + x' = SP = ST.$

The reader can easily show by geometry that if Z be the point where the tangent meets the axis of Y, SZ is at right angles to the tangent. But we shall also prove this analytically.

109. *To find the locus of the intersection of the tangent with the perpendicular drawn to the tangent from the focus.*

Let $y = mx + \frac{a}{m}$ (1) be any tangent. The equation to the line through S (that is, the point $x = a$, $y = 0$), perpendicular thereto, is $my + x = a$ (2).

Combining (1) and (2) we get the co-ordinates of Z, the foot of the perpendicular. If xy be this point Z, equations (1) and (2) state each a fact about xy; and each fact depends on the individuality of the tangent, for if we take another tangent we take another m, and other equations instead of (1) and (2). But if we eliminate m we deduce a fact about xy which is quite independent of the individual tangent, and is the equation to the locus of xy.

Now from (1) $my = m^2x + a$, and from (2) $my = a - x$.

Therefore $m^2x + a = a - x$, or $x(1 + m^2) = 0$, or $x = 0$.

Thus the x of the foot of the perpendicular is always zero: therefore the locus is the tangent at the vertex of the parabola.

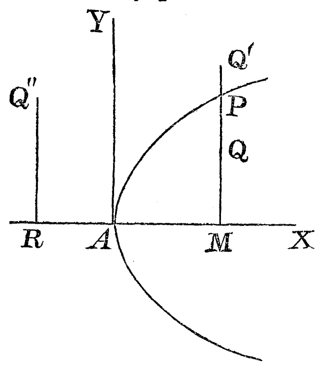
110. Let $SP = r$, $SZ = p$. Then SZ bisects the angle TSP,

$$\cos ASZ = \cos ZSP \text{ or } \frac{a}{p} = \frac{p}{r},$$

or
$$p^2 = ar.$$

Of course this could be proved by Art. 46.

111. The $y^2 - 4ax$ of any point on the curve vanishes. I



a point be within the curve, its $y^2 - 4ax$ is zero. For let QM be ordinate of such a point, and let QM meet the curve in P.

Then QM is less than PM, and the points P, Q have the same x: thus the y^2-4ax of Q is less than that of P, i.e. less than zero, and therefore negative. If the point be outside, as Q', the y^2-4ax is greater than that of P and therefore positive. At Q'' x is negative and therefore y^2-4ax is positive. Thus the point xy is within or without the curve according as y^2-4ax is negative or positive.

112. *From an external point two tangents can be drawn to a parabola.*

Let hk be the external point, and let $x'y'$ be the point of contact of a tangent through hk. Then

$$ky' = 2a(h+x') = 2a\left(h+\frac{y'^2}{4a}\right),$$

a quadratic for finding y'. The roots are both real since k^2-4ah is positive. To each y' corresponds a $\frac{y'^2}{4a}$, or, an x'.

The equation to the chord of contact is $yk = 2a(x+h)$. (Arts. 86—90, 92, 100 may be slightly altered so as to suit the parabola.)

113. A *diameter* of a curve is the locus of the middle points of a system of parallel chords.

For example, in the circle a diameter is, according to this definition, a straight line drawn through the centre, and is perpendicular to the direction of the chords.

To find the diameter in the parabola $y^2 = 4ax$ *to the chords drawn in direction* $[l, m]$.

Let $\frac{x-x'}{l} = \frac{y-y'}{m} = r$ be one of these chords, $x'y'$ being its middle point.

At the points where this intersects the curve we have

$$x = x' + lr,\ y = y' + mr,\ \text{and}\ y^2 = 4ax.$$

Thus the equation for finding r is

$$(y' + mr)^2 = 4a(x' + lr),$$

or

$$m^2r^2 + 2r(my' - 2al) + y'^2 - 4ax' = 0.$$

The values of r are equal and of opposite sign; therefore

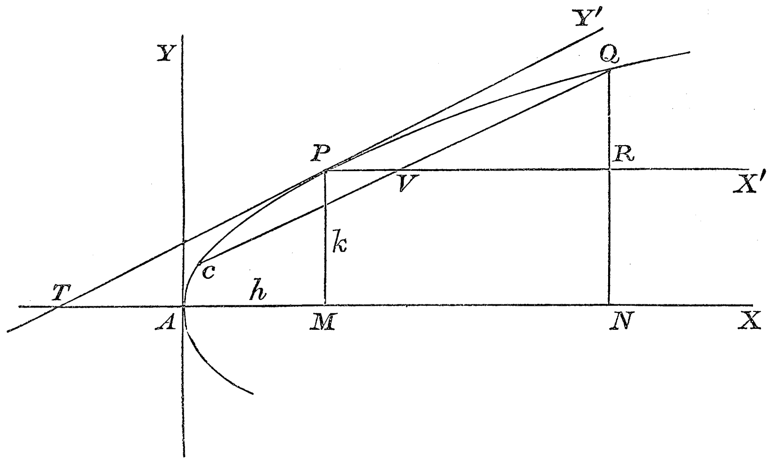
$$my' - 2al = 0.$$

Thus the locus of $x'y'$ is the straight line $my = 2al$. (See also Ex. 14.) Thus all diameters are parallel to the axis. The co-ordinates of the extremity of the diameter $my = 2al$ are

$$y = \frac{2al}{m}, \quad x = \frac{2al^2}{m^2}.$$

The reader can prove that *the tangent at this point is parallel to the chords which the diameter bisects.*

114. *To find the equation of the parabola referred to any diameter and the tangent at the extremity of that diameter.*



The equation to the parabola referred to the diameter and tangent at A is

$$y^2 = 4ax \quad \text{.........................} \quad (1).$$

We shall now change the axes to the diameter and tangent at the point P, whose co-ordinates are h, k.

Let Q be any point on the curve, x, y its co-ordinates AN, QN referred to AX, AY; x', y' its co-ordinates PV, QV referred to Ax, Ay. Let QN meet Px in R, and let the angle $yPx = \omega$.

Then $x = AN = AM + PV + VR = h + x' + y' \cos\omega$,

$$y = QN = PM + QR = k + y' \sin\omega.$$

Thus (1) is transformed to

$$(k + y' \sin \omega)^2 = 4a\,(h + x' + y' \cos \omega) \ldots\ldots\ldots\ldots\ (2).$$

Now $k^2 = 4ah$, and, as $\tan \omega = \frac{PM}{MT} = \frac{k}{2h}$, the coefficient of y' is zero. Thus (2) becomes

$$y'^2 = \frac{4a}{\sin^2 \omega}\, x', \text{ or, accents suppressed,}$$

$$y^2 = \frac{4a}{\sin^2 \omega}\, x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

Also $\operatorname{cosec}^2 \omega = 1 + \cot^2 \omega = 1 + \frac{4h^2}{k^2} = 1 + \frac{h}{a} = \frac{SP}{a}$ (Art. 108).

Thus (3) may be written

$$y^2 = 4a'x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4),$$

if a' denote the focal distance of the new origin.

115. We shall now indicate another method of obtaining this equation.

The equation required must be of the second degree, and is therefore included in the form

$$ax^2 + by^2 + c + 2a'y + 2b'y + 2c'xy = 0 \ldots\ldots\ldots\ (1).$$

The origin being on the curve, $c = 0$. Also (Art. 100) the terms of lowest dimensions vary as x, because the axis of y is a tangent at the origin. Thus $a' = 0$, and (1) is now reduced to

$$ax^2 + by^2 + 2b'x + 2c'xy = 0 \ldots\ldots\ldots\ldots\ldots\ (2).$$

For every positive value of x there are two values of y, equal in magnitude and of opposite sign: thus $c' = 0$. And for every value of y there is but one finite value of x: thus $a = 0$.

The equation is now reduced to $by^2 + 2b'x = 0$, or

$$\frac{y^2}{x} = \text{a constant}.$$

What constant is known by calculating the $\frac{y^2}{x}$ of the point A. This is found by Geometry to be $4SP$.

116. The equation to the tangent at $x'y'$ is now

$$yy' = 2a'(x + x').$$

The tangents at $(x'y')$, $(x', -y')$ meet the axis of x in the same point, or, tangents at the extremities of any chord of a parabola meet in the diameter which bisects the chord.

117. Polar Co-ordinates. (See figure to Art. 104.) *To find the equation to the parabola, the focus being pole.*

Let SP, the angle OSP, and the semilatus-rectum OS be denoted by r, θ, l. Then

$$SP = PN = OM = OS + SM = OS + SP \cos PSM,$$

or $$r = l - r\cos\theta.$$

Thus the equation to the curve is

$$\frac{l}{r} = 1 + \cos\theta \quad \text{(1)}.$$

If MSP had been the vectorial angle the equation would have been

$$\frac{l}{r} = 1 - \cos\theta \quad \text{(2)}.$$

Examples on Chapter VIII.

1. Find the equation to the parabola whose focus is the origin and directrix the line $x + y = 1$.

2. Find the equations to the tangents and normals at the extremities of the latus-rectum of $y^2 = 4ax$.

3. The tangent at any point meets the directrix and latus-rectum produced in two points equidistant from the focus.

4. Trace the curves $x^2 = 4ay$, $y^2 + 4ax = 0$, and find the equation to their common chord.

5. Find also the angles at which the curves intersect.

6. Prove by means of the form $y = mx + \frac{a}{m}$ that two tangents can be drawn to a parabola from an external point.

7. If $y = m_1x$, $y = m_2x$ be directions of tangents to $y^2 = 4ax$ from an external point $x'y'$, then

$$m_1 + m_2 = \frac{y'}{x'}, \text{ and } m_1m_2 = \frac{a}{x'}.$$

8. The directrix is the locus of points from which tangents can be drawn at right angles.

9. The directrix is the polar of the focus, and the polar of any point on the directrix is at right angles to the line joining that point to the focus.

10. What is the focal distance of the point $(-2, 4)$ in the parabola $y^2 = x$? Write down the equation to the normal at this point, and find its inclination to the axis.

11. In the curve $y^2 = -3x$ find the points at which the tangents are inclined at 30° to the axis, and prove that the focal distance of each point is $\sqrt{3}$.

12. From the vertex of $y^2 = 4ax$ a perpendicular is drawn to the tangent. Prove that the locus of the foot has for its equation

$$x(x^2 + y^2) + ay^2 = 0.$$

13. If the perpendicular be drawn to the *normal*, the locus is

$$y^2(x^2 + y^2 - 2ax) = ax^3.$$

14. Find the ordinates of the points where $y = mx + c$ meets the parabola, and prove that if m be given and c vary, the locus of the middle point of the chord is the straight line

$$y = \frac{2a}{m}.$$

15. Is the point $(1, -3)$ within or without the parabola $x^2 + 2y = 0$?

16. Draw the curves $(y-3)^2 = 4(x+2)$, $(y+3)^2 = -4(x-2)$.

17. Find the equation to the parabola $y^2 = 3x$ referred to the diameter and tangent at an extremity of the latus-rectum.

18. Refer $y^2 = 4ax$ to the tangents at the extremities of the latus-rectum.

19. If PT be the tangent at P and PM an ordinate to the diameter TM, TM is bisected by the curve.

20. Give a geometrical construction for the polar of any point.

21. Find the ratio in which the line joining xy and $x'y'$ is divided by the curve $y^2 = 4ax$, and hence shew that the equation to the two tangents from $x'y'$ is

$$(y^2 - 4ax)(y'^2 - 4ax') = \{yy' - 2a(x + x')\}^2.$$

22. Find the polar equation to the parabola, the vertex being pole.

23. From the vertex are drawn two chords AP, AQ at right angles. Find the least area of the triangle APQ.

24. Find the polar equation to the parabola referred to an extremity of the latus-rectum as pole and the latus-rectum as initial line.

25. The focus being origin, the equation to any tangent can be put in the form

$$y = m(x + a) + \frac{a}{m} \cdot$$

26. From no point can there be drawn more than three normals to a parabola.

27. If the normals to $y^2 = 4ax$ at x_1y_1, x_2y_2, x_3y_3 meet in a point, then $y_1 + y_2 + y_3 = 0$.

28. The lines from the vertex to the points of contact of tangents from (h, k) are represented by the equation

$$hy^2 = 2x (ky - 2ax).$$

29. Determine the locus of the middle points of a focal chord.

30. The semilatus-rectum is a harmonic mean between the segments of any focal chord.

31. A point moves so that its distance from one given line varies as the square of its distance from another given line. Prove that the locus of the point is a parabola having the first line for a tangent and the second for the corresponding diameter.

32. A parabola has a given axis $y = 0$ and passes through a given point $(0, k)$. Prove that its equation is of the form

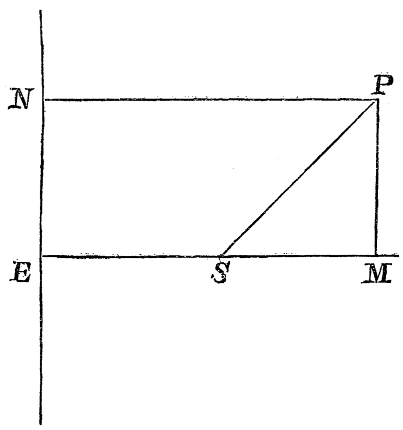
$$y^2 - k^2 = k^2 . \frac{x}{h},$$

and examine the case in which h is infinite.

CHAPTER IX.

THE ELLIPSE.

118. From S the focus of any conic section draw SE perpendicular to the directrix EN, and take ES, EN for axes of x and y. Let e be the eccentricity of the curve, and x, y the



co-ordinates PN, PM of any point P on the curve. Also let $ES=p$ (so that in the case of a parabola p is the semilatus-rectum). Then

$$SP=e\,.\,PN;\quad\text{or}\quad SM^2+MP^2=e^2\,.\,PN^2;$$

or
$$(x-p)^2+y^2=e^2x^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

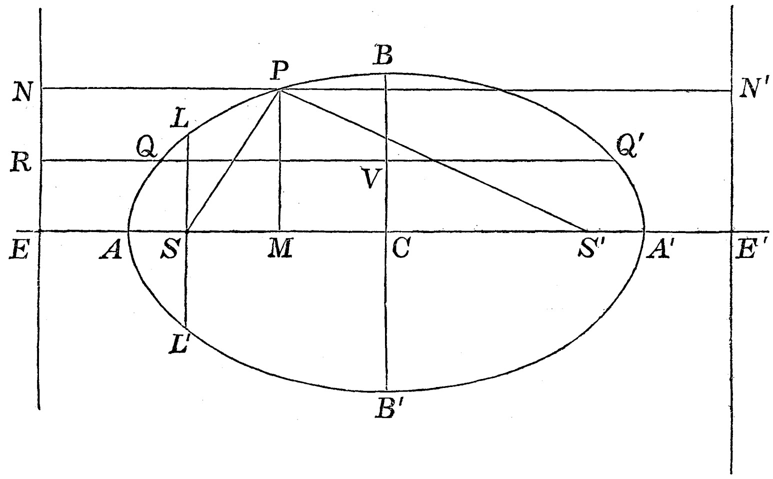
This is the equation to the conic section.

For every value given to x in (1) there are two values of y, equal in magnitude, but contrary in sign. Thus, the curve is 'symmetrical with respect to the line' ES.

119. The case in which $e=1$ has been treated in the last chapter. We now take the case in which $e<1$. The curve is an ellipse.

In the equation $(x-p)^2+y^2=e^2x^2$ (1) make $y=0$; then

$$x=\frac{p}{1\pm e}.$$



That is, if A, A' be the points where the curve meets FS,

$$EA=\frac{p}{1+e},\quad EA'=\frac{p}{1-e}.$$

Also
$$AA'=\frac{2ep}{1-e^2}.$$

If C be the middle point of AA',

$$AC=\frac{ep}{1-e^2},$$

and
$$EC=\frac{1}{2}(EA+EA')=\frac{p}{1-e^2}=\frac{AC}{e}.$$

Also $SC = \frac{1}{2}(SA' - SA) = \frac{1}{2}(e\,.\,EA' - e\,.\,EA) = e\,.\,AC.$

Thus $CS\,.\,CE = CA^2$.

Again, in (1) make $x = EC$ or $\frac{p}{1-e^2}$; then $y^2 = \frac{e^2p^2}{1-e^2}$. That is, if B, B' be the points where the curve meets an ordinate through C,

$$CB^2 = CB'^2 = AC^2(1-e^2).$$

Again, let LL' be the double ordinate through S. Then $LS = e\,.\,ES$; that is, the semilatus rectum is

$$ep = AC(1-e^2) = \frac{BC^2}{AC}.$$

120. In equation (1) of Art. 119, give y any value ER. Then the sum of the values RQ, RQ' of x is, by the theory of quadratic equations, $\frac{2ep}{1-e^2}$. Thus $\frac{1}{2}(RQ + RQ') = EC$, or the middle point V of any chord QQ' parallel to AA' lies in BB'.

Thus the curve is symmetrical with respect to BB', and if in EC produced we take S' and E' such that $E'C = EC$ and $S'C = SC$, and draw $E'N'$ parallel to EN meeting NP produced in N', and join $S'P$, then S' and $E'N'$ are a focus and directrix with which the curve could have been described, and

$$S'P = e\,.\,PN'.$$

The lines AA', BB' are called the *axes* of the curve; AA' the *major* or *transverse* axis, and BB' the *minor* or *conjugate* axis. The point C is called the *centre* of the curve, and A, A' are called *vertices* of the curve. Every chord through C is bisected at C. This is clear from the double symmetry of the curve.

Let $AA' = 2a$, $BB' = 2b$. Then

$$a^2 = b^2(1-e^2), \quad CS = ae, \quad CE = \frac{a}{e}, \quad SL = \frac{b^2}{a} = a(1-e^2).$$

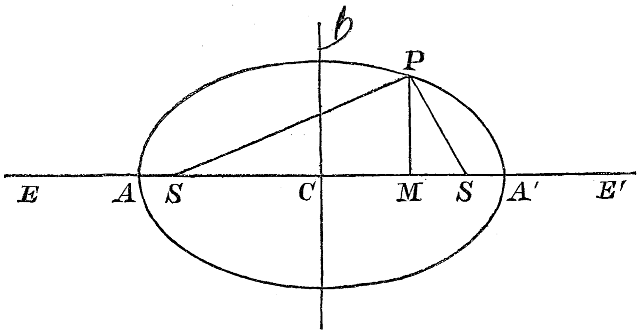
121. In the equation to the curve write $x + \frac{p}{1-e^2}$ for x.

The axes of co-ordinates are now CA' and CB, and the equation is

$$\left(x+\frac{e^2p}{1-e^2}\right)^2+y^2=e^2\left(x+\frac{p}{1-e^2}\right)^2;$$

or
$$x^2(1-e^2)+y^2=\frac{e^2p^2}{1-e^2}=a^2(1-e^2);$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



This equation shows that the ellipse is bounded in every direction, for if x^2 exceed a^2, y^2 is negative, and if y^2 exceed b^2, x^2 is negative. The same appears from the polar form

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}.$$

122. Let PM be the y of any point xy on the curve. Then the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ or } \frac{y^2}{a^2-x^2}=\frac{b^2}{a^2}$$

asserts that

$$\frac{PM^2}{AM.A'M}=\frac{BC^2}{AC^2}.$$

We may observe that

$$SP=e.ME=e(CE+CM)$$
$$=a+ex,$$

and
$$S'P=e.ME'=e(CE'-CM)=a-ex.$$

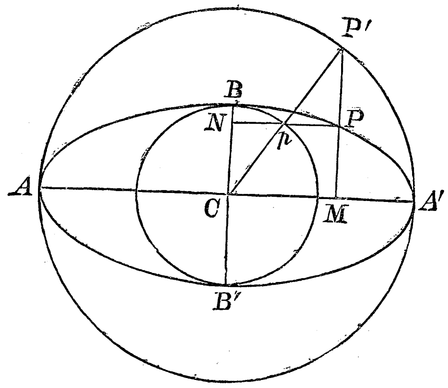
The sum of the focal distances is the same for all points on the curve, namely $2a$.

123. The equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

suggests the Trigonometrical formula

$$\sin^2\phi + \cos^2\phi = 1.$$



On AA', BB' describe circles (*auxiliary* circles they are called), and from P the point xy draw ordinates PM, PN meeting these circles in P', p. Join CP'. Then $CM = CP' \cos P'CA'$, or $x = a \cos P'CA'$. Let the angle $P'CA'$ be denoted by ϕ. Then $x = a\cos\phi$, and from (1) $y = b\sin\phi$.

Thus $$\frac{PM}{P'M} = \frac{b\sin\phi}{a\sin\phi} = \frac{b}{a}.$$

At the point where CP' meets the smaller circle

$$y = b\cos BCP' = b\sin\phi.$$

This point therefore lies in PN and is the point p.

Thus $$\frac{PN}{pN} = \frac{a\cos\phi}{b\cos\phi} = \frac{a}{b}.$$

The angle ϕ is called the *eccentric angle* of the point P, and CpP' the *eccentric radius vector* of the point P.

124. If $x'y'$, $x''y''$ be two points on the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

then
$$\frac{x'^2-x''^2}{a^2}+\frac{y'^2-y''^2}{b^2}=0,$$

or
$$\frac{x'-x''}{y'-y''}=-\frac{a^2}{b^2}\cdot\frac{y'+y''}{x'+x''}.$$

Thus the equation to the chord joining the points is

$$\frac{x-x'}{a^2(y'+y'')}+\frac{y-y'}{b^2(x'+x'')}=0,$$

and the equation to the tangent at $x'y'$, found by making $x''=x'$ and $y''=y'$, is

$$\frac{xx'-x'^2}{a^2}+\frac{yy'-y'^2}{b^2}=0,$$

or
$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This can of course be obtained by the method of Art. 100.

The normal is perpendicular to the tangent, and therefore its equation is of the form

$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=\text{constant}.$$

And as the normal contains $x'y'$ the constant is

$$\frac{a^2x'}{x'}-\frac{b^2y'}{y'}$$

or a^2-b^2. Thus the normal's equation is

$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

See Examples 8, 9.

125. The equation to the tangent can be expressed in terms of the tangent's inclination to the axis of x. For the equation is

$$y=-\frac{b^2x'}{a^2y'}\cdot x+\frac{b^2}{y'}.$$

And if m denote $-\dfrac{b^2x'}{a^2y'}$, $\left\{\text{so that } m^2 = \dfrac{b^4}{a^2y'^2}\left(1 - \dfrac{y'^2}{b^2}\right) \text{ and}\right.$ therefore $\left.\dfrac{b^4}{y'^2} = a^2m^2 + b^2\right\}$, this becomes

$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

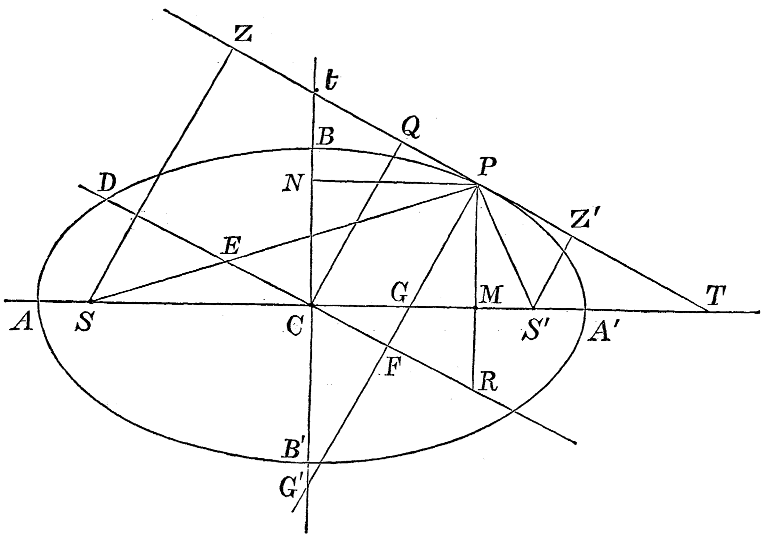
The sign is doubtful, because *two* tangents can be drawn in a given direction to an ellipse. (Compare Chap. VI. Ex. 15).

Similarly, the equation to the normal can be put in the form

$$y = mx \pm \frac{(a^2 - b^2)\,m}{\sqrt{a^2 + b^2m^2}}.$$

126. In the equation $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} = 1$ put $y = 0$. Then $xx' = a^2$; that is, if the tangent PT at P meet CA' in T, and PM be P's ordinate,

$$CM \,.\, CT = a^2.$$



Similarly, if TP meet CB in t, and PN be perpendicular to CB,

$$CN \,.\, Ct = b^2.$$

Let the normal meet the axes in G, G'.

In the equation $\dfrac{a^2x}{a'} - \dfrac{b^2y}{y'} = a^2 - b^2$ make $y = 0$. Then

$$x = \frac{a^2 - b^2}{a^2} x' = e^2x', \text{ or } CG = e^2 . CM.$$

Make $x = 0$. Then $-y = \dfrac{a^2 - b^2}{b^2} y'$, or $CG' = e^2 . \dfrac{a^2}{b^2} . CN.$

Also $$SG = ae + e^2x = e . SP,$$

and $$S'P = ae - e^2x = e . S'P.$$

Therefore (Euclid, VI. 3) *the normal bisects the angle between the focal distances.* The tangent bisects the supplemental angle.

127. *To find the locus of the foot of the perpendicular from the focus on the tangent.*

If $$y - mx = \sqrt{a^2m^2 + b^2} \quad \text{.....................} (1)$$

be any tangent, the perpendicular SZ from S is

$$my + x = ae \quad \text{........................} (2).$$

At the point Z, (1) and (2) are both true; therefore at Z,

$$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2,$$

or, $$(x^2 + y^2)(1 + m^2) = a^2m^2 + a^2,$$

or, $$x^2 + y^2 = a^2.$$

Thus Z (and similarly Z') lies on the greater auxiliary circle.

128. The equation to the tangent can be expressed in terms of the tangent's distance from the centre, and the inclination of this distance to the axis. For let

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{..................} (1)$$

be identical with

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0 \quad \text{..................} (2).$$

Then

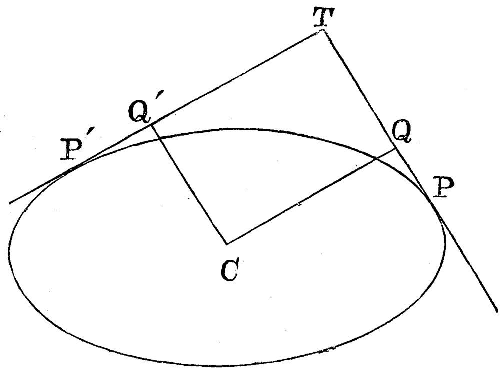
$$\frac{\frac{x'}{a}}{a\cos\alpha}=\frac{\frac{y'}{b}}{b\cos\alpha}=\frac{1}{p}=\frac{\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}}{\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}}=\frac{1}{\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}}.$$

Thus $$p=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha},$$

and (2) becomes $x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}.$

129. An important result can be obtained by Art. 128.

Let T be the intersection of tangents TP, TP' which cut at right angles.



Draw the perpendiculars CQ, CQ', and let their inclinations to the axis be α, $\frac{\pi}{2}+\alpha$. Then

$$CQ^2=a^2\cos^2\alpha+b^2\sin^2\alpha,$$

and $$CQ'^2=a^2\sin^2\alpha+b^2\cos^2\alpha.$$

Therefore $CQ^2+CA'^2$ or $CT^2=a^2+b^2$, a constant.

Thus the locus of the intersection of tangents at right angles is a circle concentric with the ellipse, and of radius

$$\sqrt{a^2+b^2}.$$

This circle, by analogy with the directrix of the parabola, is called the *director-circle* of the ellipse (see Chap. VIII. Ex. 8).

130. The $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ of any point is positive, zero, or negative, according as the point is without, on, or within the ellipse.

It may be shewn (compare Art. 85) that from any external point can be drawn two tangents. The chord of contact corresponding to $x'y'$ is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

Arts. 87—90, 92, 100, 101 can be slightly modified so as to suit the ellipse.

131. *To find the diameter conjugate to chords in direction* [l, m].

Let
$$\frac{x-x'}{l}=\frac{y-y'}{m}=r$$

be the equations to one of the chords, $x'y'$ being the middle point. At any point of the chord,

$$x=x'+lr, \quad y=y'+mr;$$

and at the points where the chord and curve intersect

$$\left(\frac{x'+lr}{a}\right)^2+\left(\frac{y'+mr}{b}\right)^2=1.$$

The values of r in this quadratic are equal and of opposite sign. Therefore

$$\frac{lx'}{a^2}+\frac{my'}{b^2}=0.$$

Thus the locus of $x'y'$ is the line

$$\frac{lx}{a^2}+\frac{my}{b^2}=0 \quad\text{......................(1)}$$

which passes through the centre of the ellipse.

By varying the ratio of $l:m$ we can make the coefficients in (1) have any ratio we please. Thus, *every* central chord is a diameter.

It l, m be only *proportional* to direction-cosines, equation (1)

is not altered. Let (l', m') be in this general case the direction of the line (1). Then l', m' are proportional to

$$\frac{a^2}{l}, \; -\frac{b^2}{m},$$

that is,
$$\frac{ll'}{a^2} + \frac{mm'}{b^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

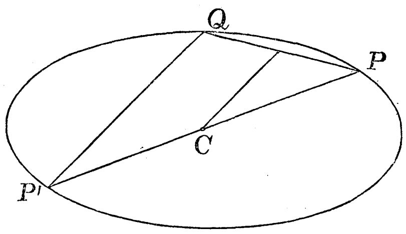
The symmetry of (2) proves that if a diameter in direction (l', m') bisect the chords in direction (l, m), then the diameter in direction (l, m) bisects the chords in direction (l', m'). Two such diameters are called *conjugate.*

Any line parallel to one diameter is conjugate to any line parallel to the other. In short, the directions (l, m), (l', m') may be called conjugate, and (2) is the condition for this relation.

The diameters $y = mx$, $y = m'x$ are conjugate if

$$mm' = -\frac{b^2}{a^2}.$$

132. From any point Q on the curve draw QP, QP' to the extremities of a diameter. These chords are called *supplemental.*



The line from the centre to the middle point of QP bisects both QP and $P'P$, and is therefore parallel to QP'. Thus, supplemental chords are also conjugate.

133. The tangent at an extremity of a diameter, being an extreme chord of the conjugate system, is parallel to the conjugate diameter. (See Art. 83.) Let PP', DD' be conjugate diameters: then the tangents at P, P', D, D' form a parallelo-

gram circumscribing the ellipse. The normals at P, P' are perpendicular to D, D'.

134. *Given* x', y' *the co-ordinates of an extremity of a diameter, to find* x'', y'', *those of an extremity of the conjugate diameter.*

Since the lines $\frac{x}{x'} = \frac{y}{y'}$, $\frac{x}{x''} = \frac{y}{y''}$ are conjugate, by Art. 131

$$\frac{x'x''}{a^2} + \frac{y'y''}{b^2} = 0.$$

Thus
$$\frac{\frac{x''}{a}}{\frac{a}{x'}} = -\frac{\frac{y''}{b}}{\frac{b}{y'}} = \frac{\sqrt{\frac{x''^2}{a^2} + \frac{y''^2}{b^2}}}{\sqrt{\frac{a^2}{x'^2} + \frac{b^2}{y'^2}}} = \pm\frac{x'y'}{ab},$$

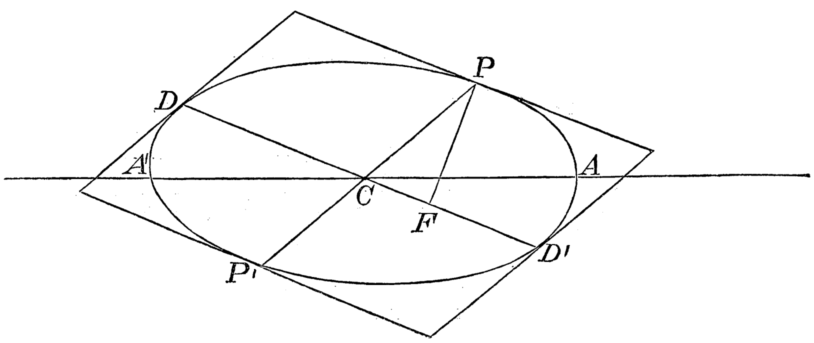
$$\text{or} \quad x'' = \pm\frac{ay'}{b}, \quad y'' = \mp\frac{bx'}{a}.$$

Cor. The eccentric angles of the points $x'y'$, $x''y''$ differ by a right angle.

135. The sum of the squares of two semi-conjugate diameters is constant, for

$$x'^2 + y'^2 + x''^2 + y''^2 = x'^2\left(1 + \frac{b^2}{a^2}\right) + y'^2\left(1 + \frac{a^2}{b^2}\right) = a^2 + b^2.$$

136. Also the parallelogram in Art. 133 is of constant area.



For draw PF at right angles to DD'. Then PF = the perpendicular from C on the tangent at P

$$=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha},$$

if α be the inclination to CA' of this perpendicular (Art. 128).

Also, as $\frac{\pi}{2}+\alpha$ is D's vectorial angle,

$$\frac{1}{CD^2}=\frac{\sin^2\alpha}{a^2}+\frac{\cos^2\alpha}{b^2} \text{ (Art. 121).}$$

Thus the area $=4PF\,.\,CD=4ab$.

137. The diameters $y=mx$, $y=-mx$ are conjugate if

$$mm'=-\frac{b^2}{a^2}.$$

Let m, m' be numerically equal, then these conjugate diameters, being equally inclined to AA', are equal. And since m is $\pm\frac{b}{a}$, they are the diagonals of the rectangle formed by tangents at A, A', B, B'.

The length of an equi-conjugate radius vector is $\sqrt{\frac{a^2+b^2}{2}}$ (Art. 135).

138. Let SP, CD meet in E (see fig. to Art. 126); PM, CD in R; and let $SZ=\varpi$, $S'Z'=\varpi'$, $SP=r$, $SP'=r'$.

It can be shewn geometrically that

$$\varpi\varpi'=b^2 \text{ .... (1),}$$

and that PE is equal and parallel to CZ', and that therefore

$$PE=a \text{ .... (2).}$$

Then as the angles SPZ, $S'PZ'$, PEF are equal, their sines are equal. Thus

$$\frac{\varpi}{r}=\frac{\varpi'}{r'}=\frac{PF}{a}=\frac{b}{CD} \text{ (Art. 136);}$$

therefore as $\varpi\varpi' = b^2$,

$$rr' = CD^2 \text{..........................} (3),$$

which therefore $= a^2 - e^2x^2$, and

$$\varpi^2 = b^2 \frac{r}{r'}, \quad \varpi'^2 = b^2 \frac{r'}{r} \text{....................} (4).$$

Again, from the nature of the quadrilateral $GFRM$,

$$\begin{aligned} PF \,.\, PG &= PM \,.\, PR \\ &= CN \,.\, Ct \\ &= b^2 \text{..............................} (5). \end{aligned}$$

Similarly $$PF \,.\, PG' = a^2 \text{........................} (6).$$

These results may also be obtained by analysis.

139. *To find the equation to the ellipse referred to two conjugate diameters as co-ordinate axes.*

The equation may be arrived at by the formulæ of transformation (see Chap. III. Ex. 25). But we shall proceed otherwise.

Let a', b' be the lengths of CP, CD, the semi-conjugates which are to be the axes. Assume for the equation

$$Ax^2 + By^2 + C + 2Dy + 2Ex + 2Fxy = 0 \text{.........} (1),$$

(a legitimate assumption, by Art. 103).

For each value of x between a and $-a$ the two values of y are to be equal and of opposite sign. Thus, for an infinite number of values of x, $2D + 2Fx$ vanishes. Therefore $D = 0$ and $F = 0$. Similarly $E = 0$.

We have now reduced (1) to

$$Ax^2 + By^2 + C = 0,$$

$$\text{or } \frac{x^2}{-\frac{C}{A}} + \frac{y^2}{-\frac{B}{A}} = 1 \text{.....................} (2).$$

Make $y = 0$; then x^2, or a'^2, $= -\frac{C}{A}$.

Make $x = 0$; then y^2, or b'^2, $= -\dfrac{B}{A}$.

Thus the equation sought for is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

140. The equation to the tangent at $x'y'$ (or to the polar of $x'y'$) is now

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the directions (lm), $(l'm')$ are conjugate if

$$\frac{ll'}{a'^2} + \frac{mm'}{b'^2} = 0.$$

The reader will easily prove that tangents at the extremity of any chord meet in the conjugate diameter.

141. *To find the polar equation to the ellipse, the focus being pole.*

In the figure of Art. 119 let $SP = r$, and the angle $ASP = \theta$. Then

$$SP = e \, . \, PN = e \, . \, EM = e\,(ES + SP \cos PSM),$$

$$\text{or } r = ep - er \cos \theta.$$

Thus the equation is

$$\frac{ep}{r} = 1 + e \cos \theta,$$

$$\text{or } \frac{a\,(1 - e^2)}{r} = 1 + e \cos \theta,$$

or, if l denote the semilatus-rectum,

$$\frac{l}{r} = 1 + e \cos \theta.$$

If θ had been the angle $A'SP$, the equation would have been

$$\frac{l}{r} = 1 - e \cos \theta.$$

Examples on Chapter IX.

N. B. When nothing is implied to the contrary, it is to be understood that the equation to the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. Find the equation to the ellipse which has the point $(1, -2)$ for a focus, and the line $3x-7y=2$ for the corresponding directrix, the eccentricity being $\frac{1}{3}$.

2. What is the eccentricity of the curve

$$x^2+y^2=(lx+my+n)^2?$$

3. What are the eccentricities of the curves

$$x^2+2y^2=1, \qquad 3x^2+5y^2=11,$$

and what are the co-ordinates of their foci?

4. The equation to the ellipse referred to its vertex A as origin is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a}.$$

5. Find the equations to the tangents at the extremities of the latera recta. Where do these tangents meet AA'?

6. Two pins are fixed in a table at a distance $2c$ apart, and over them is thrown an endless string of length $2(a+c)$, which is kept tight by an upright pencil. Find in the simplest form the equation to the curve traced by the pencil on the table.

7. Find the equations to the tangent and normal at any point of the curve in Ex. 4.

8. Prove that the equation to the chord joining the points whose eccentric angles are ϕ, ϕ' is

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\frac{\phi-\phi'}{2}.$$

9. Deduce equations to the tangent and normal at one of the points.

10. Apply the form $y = mx + \sqrt{a^2m^2 + b^2}$ to find the locus of the intersections of tangents at right angles.

11. Is the point (1, 2) within or without the curve $2x^2 + y^2 = 1$?

12. Find the points of the ellipse $8x^2 + 7y^2 = 3$, where the tangents are equally inclined to the axes.

13. Find the area of the triangle TCt (fig. to Art. 126), and prove that if P be an extremity of a latus rectum,

$$\Delta TCt : \Delta SPS' :: 1 : 2e^2(1 - e^2).$$

14. A line of given length moves with its extremities on two fixed lines at right angles. Shew that the locus of any point in the line is an ellipse whose semi-axes are equal to the segments of the line.

15. Assuming that the equation $y = mx + \sqrt{a^2m^2 + b^2}$ represents a tangent to the ellipse, prove that two tangents can be drawn from an external point, and find the locus of the point when the tangents include a given angle.

16. How many normals can be drawn to an ellipse from a given point in the major axis? How many from a given point in the minor axis?

17. The line joining the centre to any point is conjugate to the polar of the point.

18. If CP meet the polar of P in Q and the curve in R, then $CP . CQ = CR^2$.

19. The directrix is the polar of the focus.

20. A chord PQ parallel to AB meets the axes in M, N. Prove that $PM = QN$.

21. In what positions of CP and CD is the angle PCD a maximum or minimum?

22. In the ellipse (see fig to Art 126) $GT \cdot CM = CD^2$.

23. If p, r be the central perpendicular and radius vector,

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - r^2}.$$

24. If p, r be the focal perpendicular and radius vector,

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

25. Prove that the lines CZ, CZ' are parallel to $S'P$, SP, and find the locus of the intersection of the diagonals of the parallelogram formed by these four lines.

26. The ellipse being referred to the equi-conjugate diameters, find the equation to the normal at any point.

27. If QV be the ordinate to any diameter PP', then

$$\frac{QV^2}{PV \cdot VP'} = \frac{CD^2}{CP^2},$$

and if the tangent at Q meet CP in T, then $CV \cdot CT = CP^2$.

28. If α, β be the direction-angles of a central perpendicular p referred to two conjugates,

$$p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta.$$

29. The circles described on SP, $S'P$ touch the larger auxiliary circle.

30. Find the co-ordinates of the middle point of the chord $y = mx + c$, and if m be given find the locus of this point. (ω).

31. Find the locus of the intersection of tangents at P and D, and the locus of the middle point of PD.

32. If CP', CD' be semi-conjugates, the triangles CPP', CDD' are equal.

33. Find the condition that two pairs of conjugate diameters may form a harmonic pencil.

34. Find the polar equation to the ellipse when the foot of the directrix is origin and the directrix the initial line.

35. The length of the focal chord parallel to CP is $\frac{2\,CP^2}{a}$.

36. The sums of the squares of the reciprocals of central radii at right angles are constant.

37. Find the equation to the common diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=c^2$.

38. Find also the equation to the common tangents and prove that, if $c^2=ab$, the common diameters bisect the common tangents.

39. Find the angle of intersection of the curves in Ex. 37 and the length of common tangent.

40. The points from which four normals can be drawn to an ellipse, two and two at right angles, lie in the equi-conjugate diameters.

41. Find the locus of the middle points of chords drawn through a fixed point.

42. When is a curve symmetrical with respect to a given line?

Prove that the curve $x^3+y^3=a^3$ is symmetrical with respect to the line $x=y$.

43. An ellipse slides between two lines at right angles. Find the locus of the centre.

44. Two ellipses equal in all respects have the same centre. Prove that their common tangents form a rectangle.

45. From the equation $y = mx + \sqrt{a^2m^2 + b^2}$ deduce the equation to the tangent to the parabola.

46. Find the loci of the points E, F, R in the fig. to Art 126.

47. P is a point on the auxiliary circle, and PA, PA' meet the ellipse in Q, Q'. Prove that

$$\frac{AP}{AQ} + \frac{A'P}{A'Q'} = 1 + \frac{a^2}{b^2}.$$

48. Given the foci, to what form does the ellipse approximate as the eccentricity increases?

49. Given the points A, A', to what form does the ellipse approximate as the eccentricity diminishes?

50. The locus of the foot of the central perpendicular is the curve

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

51. From a fixed point on an ellipse a chord is drawn, and a diameter is drawn parallel to the chord. Find the locus of the point where this diameter meets the tangent at the variable extremity of the chord.

52. Find the condition that the line $Ax + By + C$ may touch the ellipse (see Art. 128).

53. Find the condition that the line joining the points xy, $x'y'$ may touch the ellipse, and hence prove that the equation to the two tangents from $x'y'$ is

$$b^2(x - x')^2 + a^2(y - y')^2 = (xy' - yx')^2.$$

54. A right-angled triangle having its right angle at a given point is inscribed in an ellipse. Prove that the hypothenuse meets the normal at the right angle in a fixed point.

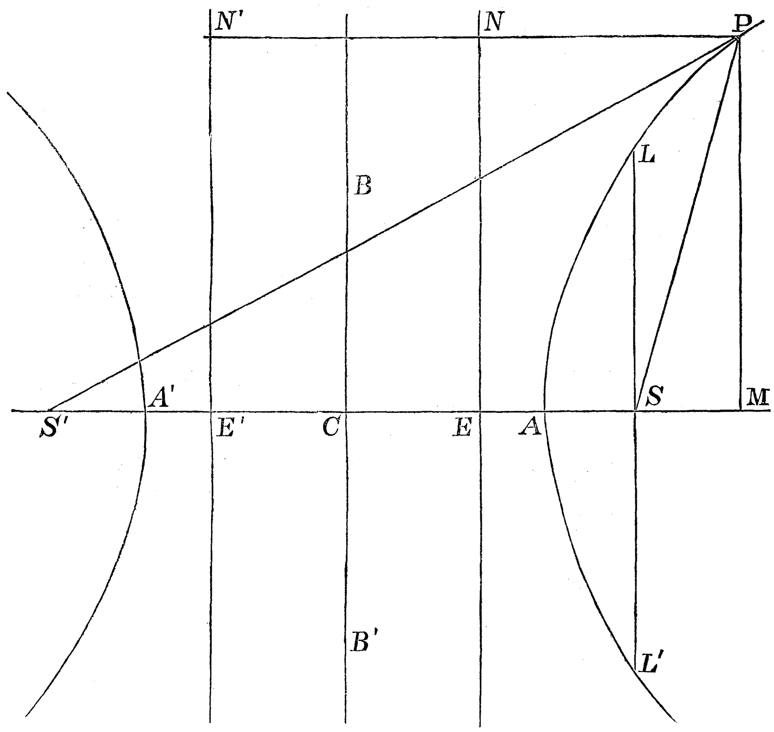
CHAPTER X.

THE HYPERBOLA.

142. We return to the equation

$$(x-p)^2+y^2=e^2x^2 \quad (1) \quad \text{(Art 118)},$$

and now take the case in which $e>1$. The curve is the hyperbola. In (1) make $y=0$: then $x=\dfrac{p}{e+1}$ or $-\dfrac{p}{e-1}$. Thus



if A, A be the points where the curve meets ES, $EA=\dfrac{p}{e+1}$

and $EA' = \frac{p}{e-1}$, A, A' lying on opposite sides of E. Also $AA' = \frac{2ep}{e^2-1}$, and if C be the middle point of AA', $AC = \frac{ep}{e^2-1}$ and $EC = \frac{1}{2}(EA' - EA) = \frac{p}{e^2-1} = \frac{AC}{e}$.

Also $\quad SC = \frac{1}{2}(SA + SA') = \frac{1}{2}(e \,.\, AE + e \,.\, AE') = e \,.\, AC$.

Thus $CS \,.\, CE = CA^2$.

Again, in (1) make $x = -EC$ or $-\frac{p}{e^2-1}$: then $y^2 = -\frac{e^2p^2}{e^2-1}$. Thus the curve does not meet in real points an ordinate through C. But it is convenient to take B and B' in this ordinate such that $CB = CB' = \sqrt{\frac{e^2p^2}{e^2-1}} = CA \,.\, \sqrt{e^2-1}$. The semi-latus rectum is ep or $AC \,.\, (1-e^2)$ or $\frac{BC^2}{AC}$.

143. For any value of y the semi-sum of the values of x is the constant $-\frac{ep}{e^2-1}$. Thus the line $x = -\frac{ep}{e^2-1}$, or BB' divides the curve symmetrically. In EC produced take E', S', such that $CE' = CE$ and $CS' = CS$, and draw $E'N$ parallel to CB. Then S' and $E'N$ are a focus and directrix by means of which the curve could have been described, and if P be any point on the curve and PNN' a parallel to AA' meeting the directrices in N, N',

$$SP = ePN \text{ and } S'P = ePN'.$$

The lines AA', BB' are the transverse and conjugate axes of the curve. The points C, A, A' are the centre and vertices. Every chord through C is bisected at C.

Let $AA' = 2a$, $BB' = 2b$. Then

$$a^2 = b^2(e^2-1), \quad CS = ae, \quad CE = \frac{a}{e}, \quad SL = \frac{b^2}{a} = a(e^2-1).$$

144. Writing $x - \frac{p}{e^2-1}$ for x we obtain for the equation to the curve referred to the centre,

$$x^2(e^2-1) - y^2 = a^2(e^2-1),$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ ....(1).}$$

When $x^2 < a^2$, y^2 is negative. Thus the curve has no real point between the lines $x = a$, $x = -a$. When x is infinite, so is y. Thus the curve consists of two infinite branches.

The polar form of (1) is

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}.$$

When $\tan^2\theta$ exceeds $\frac{b^2}{a^2}$, $\frac{1}{r^2}$ is negative. Thus the lines $\tan\theta = \pm\frac{b}{a}$, or $y = \pm\frac{b}{a}x$, or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, enclose the curve.

These lines are the diagonals of the parallelogram formed by drawing parallels to the axes through A, A', B, B'. They are called *asymptotes* of the curve.

145. Let PM be the y of any point xy on the curve. Then the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ asserts that $\frac{PM^2}{AM \cdot A'M} = \frac{BC^2}{AC^2}$.

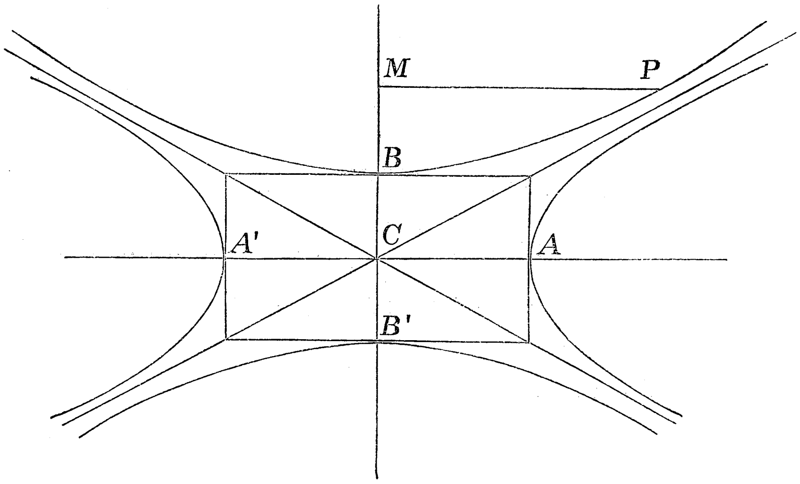
Also $\quad SP = e \cdot ME = e(CM - CE) = ex - a,$

$$S'P = e \cdot ME' = e(CM + CE') = ex + a.$$

The difference of the focal distances of any point on the curve is the constant $2a$.

146. There is no restriction as to the relative magnitude of a and b. For, as the curve can have any eccentricity from 1 to ∞, $e^2 - 1$ or $\frac{b^2}{a^2}$ can have any positive value.

When $e = \sqrt{2}$, $a = b$, and the asymptotes are at right angles. This hyperbola is called *equilateral* or *rectangular*. Its equation is $x^2 - y^2 = a^2$.



147. Let B, B' be called *false* vertices of the curve*. Then the hyperbola whose real vertices are B, B' and false vertices A, A' has the same asymptotes as the first hyperbola, for they are derived from the same rectangle. Each hyperbola has the real and false vertices of the other for its own false and real vertices. The curves are called *conjugate* hyperbolæ.

Let P be any point on the new curve. Draw PM perpendicular to CB. Then

$$\frac{CM^2}{CB^2} - \frac{PM^2}{CA^2} = 1.$$

That is, the equation to the new curve is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1;$$

and it is derived from the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by changing a^2, b^2 to $-a^2$, $-b^2$, or by changing the 1 into -1.

* The *true* vertices on the conjugate axes are imaginary, being the points $(0, b\sqrt{-1})$, $(0, -b\sqrt{-1})$.

148. The equation to the tangent at $x'y'$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \text{ .... } (1).$$

The equation to the normal is

$$\frac{a^2x}{x'} + \frac{b^2y}{y'} = 1 \text{ .... } (2).$$

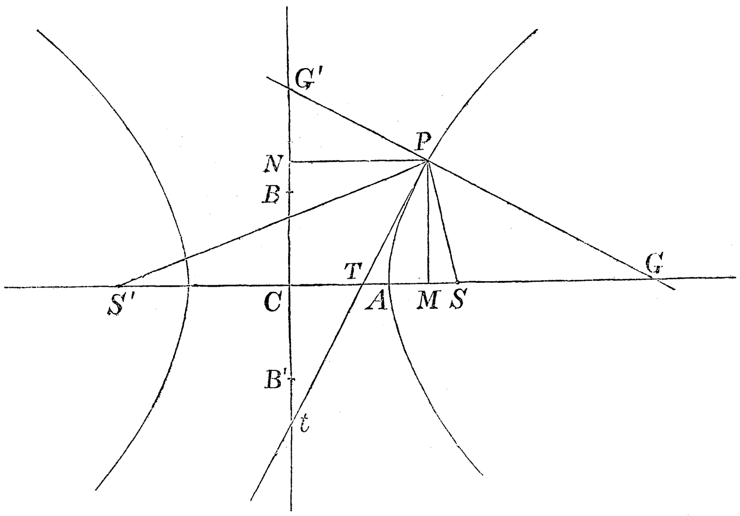
Other forms of (1) and (2) are

$$y = mx \pm \sqrt{a^2m^2 - b^2} \text{ .... } (3),$$

$$y = mx \pm \frac{(a^2 + b^2)\, m}{\sqrt{a^2 - b^2m^2}} \text{ .... } (4).$$

All these results can be obtained from Arts. 124, 125 by writing $-b^2$ for b^2.

149. As in Art. 126,



$$CM \,.\, CT = a^2, \quad CN \,.\, Ct = b^2, \quad CG = e^2 \,.\, CM, \quad CG' = e^2 \,.\, CN.$$

Also $\qquad SG = e^2x - ae = e \,.\, SP,$

and $\qquad S'G = e^2x + ae = e \,.\, S'P.$

Therefore (Euclid VI.A) the normal bisects the external angle between the focal distances. The tangent bisects the internal angle.

The locus of the foot of the perpendicular from the focus on the tangent is the circle described on AA' as diameter.

150. Corresponding to the formula

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

of Art. 128 we have

$$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha.$$

The perpendicular vanishes, or the tangent passes through the centre, when $\tan^2 \alpha = \frac{a^2}{b^2}$. The equation to the tangent is then $y = -\cot \alpha \,.\, x$ or $y = \mp \frac{b}{a} \,.\, x$. That is, the asymptotes are two tangents drawn from the centre. The chord of contact is the line at infinity, and the centre is the pole of the line at infinity.

In the ellipse and hyperbola p is always finite.

In the parabola the focal perpendicular is $\sqrt{ar}$, and when r is infinite, so is $\sqrt{ar}$. Thus *every parabola touches the line at infinity.*

The radius of the director-circle is $\sqrt{a^2 - b^2}$. The radius of the director-circle of the conjugate hyperbola is $\sqrt{b^2 - a^2}$. Of the two circles one is imaginary, except in the rectangular hyperbola, when both are reduced to a point-circle at C. The circle with centre C and radius $\sqrt{a^2 \sim b^2}$ is always the director-circle of one of the curves.

151. If a point lie within (or on the concave side of) the curve, its $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ is positive, for this is the case at a point within the curve on the axis of x. Similarly the $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ of any point outside (or on the convex side of) the curve is negative. (Cf. Arts. 56, 95.)

It follows that two tangents can be drawn from an external

point $x'y'$, the equation for determining the x of a point of contact being

$$x^2\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)-2xx'+a^2\left(1+\frac{y'^2}{b^2}\right)=0.$$

The product of the two values of x is positive or negative with

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}.$$

That is, the tangents from $x'y'$ are drawn to the same or different branches of the curve according as $x'y'$ lies or does not lie in the same angle of the asymptotes as the curve.

The chord of contact, or the polar of the point $x'y'$, has for its equation,

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1.$$

Arts. 87—90, 92, 100, 101 may be slightly modified so as to suit the hyperbola.

152. Art. 131, when b^2 is changed to $-b^2$, applies to the hyperbola. Thus the diameter conjugate to

$$\frac{x}{l}=\frac{y}{m} \text{ is } \frac{lx}{a^2}=\frac{my}{b^2},$$

and the condition that the directions (l, m), (l', m') may be conjugate is

$$\frac{ll'}{a^2}-\frac{mm'}{b^2}=0.$$

Hence conjugate diameters of a hyperbola are also conjugate diameters of the conjugate hyperbola.

Let the diameter conjugate to PP' meet the conjugate hyperbola in D, D'. Then the tangents at PP' are parallel to DD' (cf. Art. 83), and, as PP' is conjugate to DD' in the conjugate hyperbola, the tangents to that curve at D, D' are parallel to PP'.

The diameters $y=mx$, $y=m'x$ are conjugate if $mm'=\dfrac{b^2}{a^2}$.

If m be numerically less than $\dfrac{b}{a}$, m' is numerically greater than $\dfrac{b}{a}$. That is, of two conjugate diameters one meets the hyperbola,

and the other the conjugate hyperbola. Art. 132 needs no alteration except in the figure.

153. If $x'y'$ be the extremity of any diameter and $x''y''$ an extremity of the conjugate so that $x''y''$ lies on the conjugate hyperbola, then, since the lines

$$\frac{x}{x'} = \frac{y}{y'}, \qquad \frac{x}{x''} = \frac{y}{y''}$$

are conjugate,

$$\frac{x'x''}{a^2} - \frac{y'y''}{b^2} = 0.$$

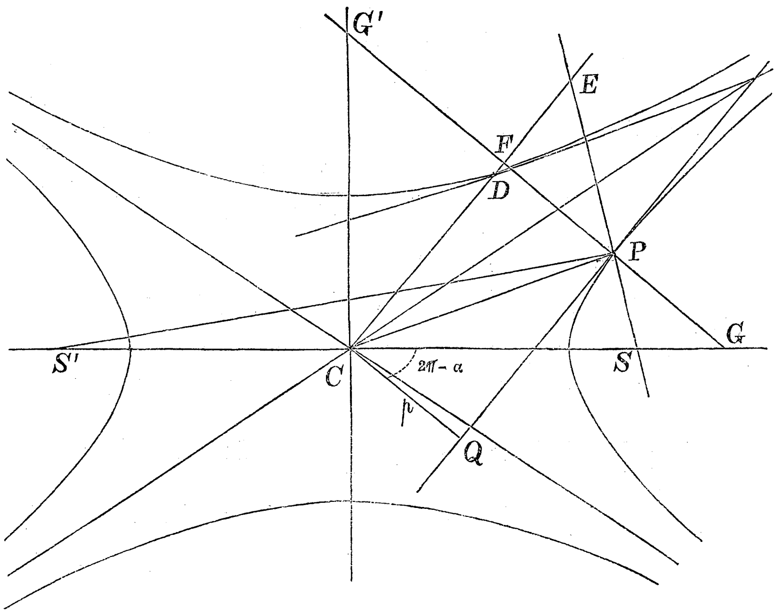
Thus

$$\frac{\frac{x''}{a}}{\frac{a}{x'}} = \frac{\frac{y''}{b}}{\frac{b}{y'}} = \frac{\sqrt{\frac{y''^2}{b^2} - \frac{x''^2}{a^2}}}{\sqrt{\frac{a^2}{x'^2} - \frac{b^2}{y'^2}}} = \pm \frac{x'y'}{ab};$$

or,

$$x'' = \pm \frac{ay'}{b}, \qquad y'' = \frac{bx'}{a}.$$

154. The difference of the squares of two semi-conjugate



diameters is the constant $a^2 \sim b^2$, and the area of the parallelogram formed by tangents at their extremities is the constant $4ab$.

In the equilateral hyperbola $a^2 \sim b^2 = 0$ and therefore all pairs of conjugate diameters are pairs of equal diameters, equally inclined to either asymptote.

155. The notation of Art. 138 being adopted, the results

$$\varpi\varpi' = b^2, \quad PE = a, \quad rr' = CD^2,$$

$$\varpi^2 = b^2 \frac{r}{r'}, \quad \varpi'^2 = b^2 . \frac{r'}{r}, \quad PF . PG = b^2, \quad PF . PG' = a^2,$$

hold for the hyperbola.

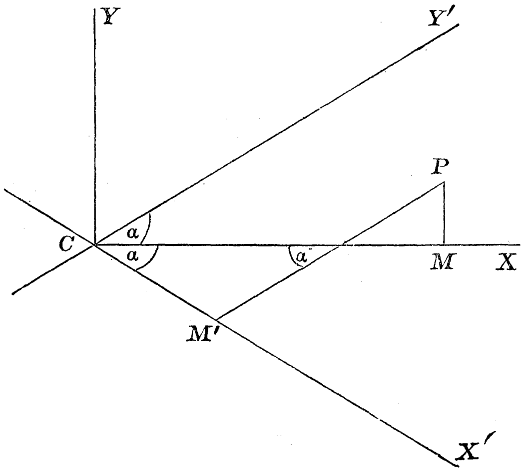
156. The equation to the hyperbola referred to CP, CD as axes, the lengths of CP, CD being a', b', is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

Thus the expression $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ has been transformed to $\frac{x^2}{a'^2} - \frac{y^2}{b'^2}$. Therefore the locus of the equation $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0$ is now what *was* the locus of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, that is, the asymptotes. Thus the asymptotes are the diagonals of the parallelogram in Art. 154, and the line PD is, by geometry, parallel to one asymptote and bisected by the other. (This line has not been drawn in the figure.)

The polar of $x'y'$ is now $\frac{xx'}{a'^2} - \frac{yy'}{b'^2} = 1$, and the directions (l, m), $(l'm')$ are conjugate if $\frac{ll'}{a'^2} - \frac{mm'}{b'^2} = 0$. The reader will easily prove that tangents at the extremities of any chord meet in the conjugate diameter.

157. *To find the equation to the hyperbola referred to the asymptotes as axes.*



Let P be any point on the curve, x, y its co-ordinates CM, MP with reference to the axes of figure, x', y' its co-ordinates CM', $M'P$ with reference to the asymptotes, and let 2α be the angle between the asymptotes.

Then $x = CM =$ projection of the broken line $CM'P$ on CX

$$= (x' + y') \cos \alpha,$$

and $y = PM =$ projection of the broken line $CM'P$ on CV

$$= (y' - x') \sin \alpha,$$

Thus the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes

$$\frac{(x' + y')^2 \cos^2 \alpha}{a^2} - \frac{(x' - y')^2 \sin^2 \alpha}{b^2} = 1,$$

or, since $\tan^2 \alpha = \frac{b^2}{a^2}$, so that $\frac{\cos^2 \alpha}{a^2} = \frac{\sin^2 \alpha}{b^2} = \frac{1}{a^2 + b^2}$,

$$4x'y' = a^2 + b^2;$$

or, accents suppressed, $xy = \frac{a^2 + b^2}{4}$.

158. This equation can be arrived at in another way.

For the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, which represents the asymptotes in the old system, is by hypothesis to become $xy = 0$.

Thus the *expression* $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ will be transformed to something varying as xy; say $\frac{xy}{\lambda}$; and the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will become

$$\frac{xy}{\lambda} = 1, \text{ or } xy = \lambda.$$

Now the xy of the point A is readily proved to be $\frac{a^2+b^2}{4}$. Thus $\lambda = \frac{a^2+b^2}{4}$, and the new equation is

$$xy = \frac{a^2+b^2}{4}.$$

The new equation to the conjugate hyperbola is $xy = -\frac{a^2+b^2}{4}$.

159. Thus the equation $xy = c^2$ represents a hyperbola, referred to its asymptotes. By varying c we get a system of different hyperbolas having the same asymptotes and lying in the same angles of the asymptotes. A particular case is that in which $c = 0$. The curve is then reduced to the two axes.

160. *To find the equation to the tangent at any point of the curve* $xy = c^2$.

The chord joining $x'y'$ and $x''y''$ [since $x'y' - x''y'' = 0$, so that $x'(y' - y'') + y''(x' - x'') = 0$] has for its equation

$$\frac{x - x'}{x'} + \frac{y - y'}{y''} = 0,$$

which becomes, when the points coincide,

$$xy' + yx' - 2x'y' = 0,$$

or

$$xy' + yx' = 2c^2 \quad \text{.........................} \quad (1).$$

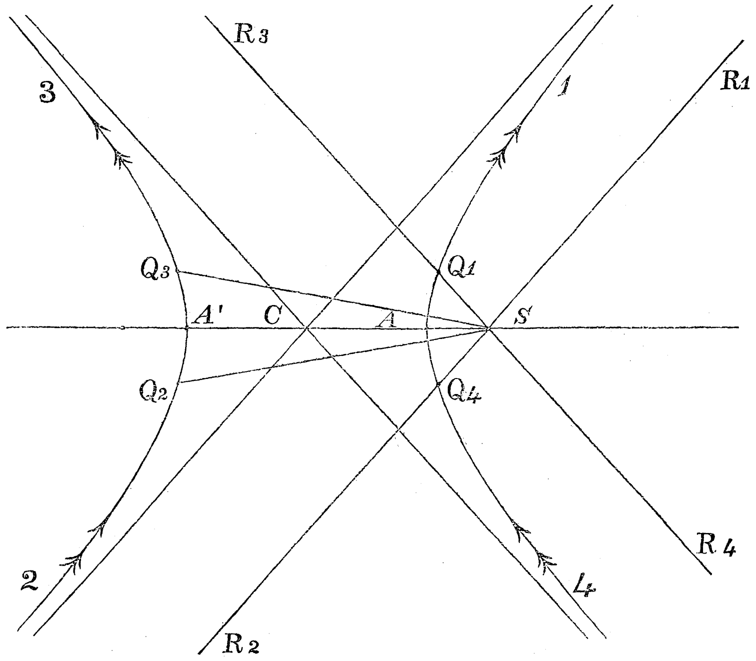
This can also be obtained by the method of Art. 100.

The intercepts of the tangent are $\frac{2m^2}{y'}, \frac{2m^2}{x'}$, or $2x', 2y'$. Thus the portion of the tangent included between the axes is bisected at the point of contact. This is evident also from geometry (see Art. 156).

161. The polar equation $\frac{l}{r} = 1 + e\cos\theta$ is the same for all conic sections. In the case of the hyperbola $l = a(e^2 - 1)$.

Through S draw lines $R_1R_2R_3R_4$ parallel to the asymptotes. These intersect the asymptotes, and therefore the curve, at infinity. And, as a straight line can meet a conic in only two points, each of these lines meets the curve in only one point at a finite distance.

Let θ, which is measured from SA towards SR, change from o to 2π. Then, since the extreme points of an asymptote are



consecutive points on the curve, the four quadrants of the curve will be described in the order indicated by the figures 1, 2, 3, 4 in the figure, and in the direction indicated by the arrows. Thus if Q_1, Q_4 be the points at a *finite* distance where R_3R_4, R_1R_2 meet the curve, the vectorial angles of these points are respectively ASQ_1 and $2\pi - ASQ_4$; and those of the corresponding points Q_2, Q_3, in the quadrants 2, 3, are $\pi - ASQ_2$ and $\pi + ASQ_3$. The radii vectores of all points on the further branch from S are negative.

162. *To find the polar equation to the chord joining the points* $\theta = \alpha + \beta$, $\theta = \alpha - \beta$, *on the conic* $\frac{l}{r} = 1 + e \cos\theta$. .... (1).

By giving the right values to A, B the equation

$$\frac{l}{r} = A\cos(\theta - B) \text{ .... } (2)$$

can be made to represent any straight line.

Suppose for a moment that the equation to the chord were to be determined in the form (2). We should combine (1) and (2), and eliminating r, give to A and B such values that the equation in θ might be satisfied by $\alpha + \beta$ and $\alpha - \beta$. But to facilitate the combination with (1) we shall assume for the equation to the chord the form

$$\frac{l}{r} = A\cos(\theta - B) + e\cos\theta \text{ ....} (3),$$

which is equally general with (2). The elimination of r from (1) and (3) gives

$$A\cos(\theta - B) = 1.$$

Therefore $A\cos(\alpha + \beta - B) = 1 = A\cos(\alpha - \beta - B)$ ... (3).

These conditions are satisfied if $\alpha + \beta - B = -(\alpha - \beta - B)$ (or $B = \alpha$) and $A = \sec\beta$.

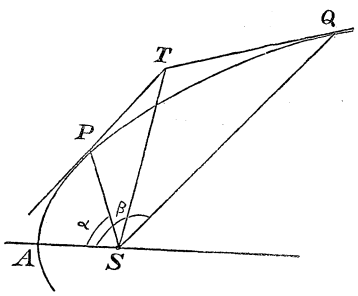
The equation to the chord is therefore

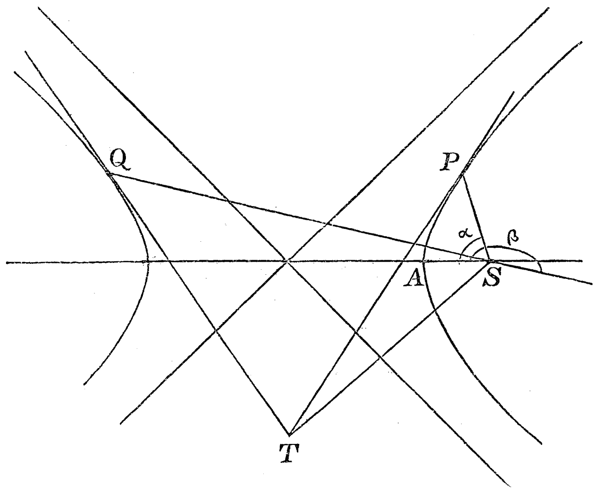
$$\frac{l}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta \text{ .... } (4).$$

By making $\beta = 0$ we deduce the equation to the tangent at the point $\theta = \alpha$, viz.

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta.$$

163. If TP, TQ be two tangents to a conic, then TS will bisect the angle PSQ, except TP, TQ be drawn to different branches of a hyperbola, in which case TS bisects the supplemental angle.





For let α, β be the vectorial angles of P, Q. Then the equations to TP, TQ are

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta, \qquad \frac{l}{r} = \cos(\theta - \beta) + e\cos\theta.$$

Combining these equations we have, at the point T,

$$\cos(\theta - \alpha) = \cos(\theta - \beta).$$

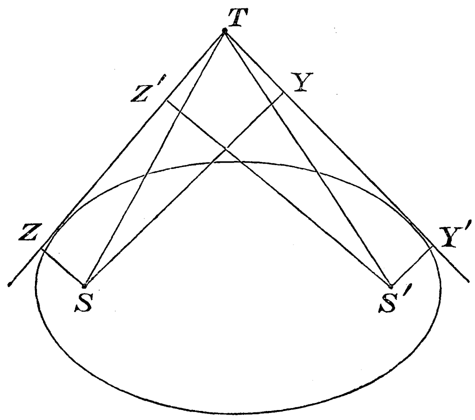
Therefore $\theta - \alpha = \beta - \theta$ (or $2\pi + \beta - \theta$);

thus $$\theta = \frac{\alpha + \beta}{2} \left(\text{or } \pi + \frac{\alpha+\beta}{2}\right),$$

that is, ST bisects the angle PSQ, with the exception stated above.

164. Again the lines ST, $S'T$ are equally inclined to the tangents.

For draw SZ, SY, SZ', SY' perpendicular to the tangents.



Then $$SZ \,.\, S'Z = BC^2 = SY \,.\, S'Y'.$$

Thus $$\frac{SZ}{SY} = \frac{S'Y'}{S'Z'}\,.$$

Let each ratio $= k$, then if the lines TP, TQ be denoted by α and β, the equations to TS, TS' will be

$$\alpha = k\beta, \qquad \alpha = \frac{\beta}{k},$$

which represent lines equally inclined to TP, TQ.

165. The parabola is the connecting link between the ellipse and hyperbola, being an extreme case of each curve. If

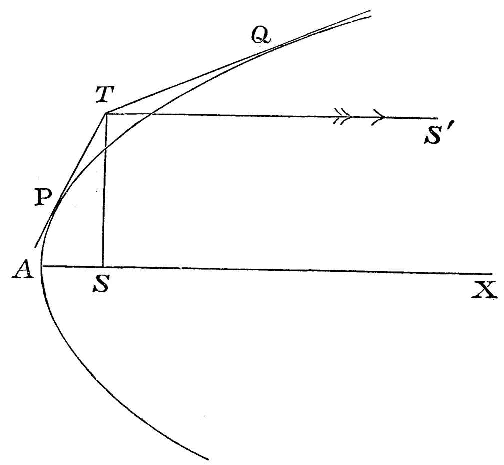
the focus and the nearer vertex of an ellipse or hyperbola be fixed (so that $a(1 \sim e)$ is given), while e is made to approach 1, the centre and further focus and vertex move off to infinity (for a becomes infinite). And the semilatus-rectum $a(1 \sim e^2)$ becomes $2a(1 \sim e)$; and the equation to the curve referred to the vertex,

or $$y^2 = (1 \sim e^2)(2ax \mp x^2),$$

becomes $$y^2 = 2ax(1 \sim e^2),$$

or $$y^2 = \text{latus rectum} \times x.$$

166. If the curve in Art. 164 be a parabola with focus S, TS' is parallel to the axis. It can of course in this case be



proved independently of Art. 164, that the angles PTS, QTS' are equal.

Examples on Chapter X.

1. Find the equation to the hyperbola which has the origin for focus and the line $x - 5y + 3$ for directrix, the eccentricity being 2.

2. Find the eccentricity and the lengths of the axes of the curve

$$(x-2)^2+(y+1)^2=3\,(x-y)^2.$$

3. What are the eccentricities of the curves

$$5x^2-3y^2+3=0,\ \ A^2x^2-B^2y^2+C=0\,?$$

4. Find the centre and the lengths of the axes of the curve

$$5x^2-2y^2+x-7y=0.$$

5. Given the base and the difference of the sides of a triangle, find in the simplest form the equation to the locus of the vertex.

6. In any hyperbola if SR be a perpendicular to an asymptote, $CR=CA$.

7. The distance, measured parallel to an asymytote, of any point on the curve from a directrix is equal to the distance from the corresponding focus.

8. Give a practical method of describing a hyperbola.

9. If e, e' be eccentricities of two conjugate hyperbolas,

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

10. Find the equation to the hyperbola which is conjugate to

$$x^2-y^2+Ax+By+C=0.$$

11. In the rectangular hyperbola $PG=PG'$.

12. PQ is any chord perpendicular to a fixed diameter AB of a given circle. Find the locus of the intersection of AP, BQ.

13. Also, assuming that the locus is a conic section, find *a priori* its transverse axis and eccentricity.

14. Find the condition that the line $Ax + By + C$ may touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1. \quad \text{(Art. 150.)}$$

15. Find the condition that the line joining xy, $x'y'$ may touch the curve, and deduce the equation to the two tangents from $x'y'$, viz.

$$\left(\frac{xx'}{a^2} - \frac{yy'}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1\right).$$

16. Is the point $(-3, 5)$ on the convex or on the concave side of the hyperbola

$$7y^2 - 3x^2 = 2\,?$$

17. The polars of the same point with respect to two conjugate hyperbolas are parallel.

18. A point moves so that its polars with respect to two conjugate hyperbolas are a constant distance apart. Prove that the locus of this point is an ellipse, whose shape and position are independent of the constant distance.

19. Find by transformation of co-ordinates the equation to the hyperbola referred to two conjugate diameters.

20. The asymptotes and a pair of conjugate diameters form a harmonic pencil.

21. Are the asymptotes a pair of conjugate diameters?

22. Find the equation to the normal at any point of the curve

$$xy = c^2 \;\ldots\ldots\; (\omega).$$

23. Express in polar co-ordinates the equations to the tangent and normal at any point of the curve

$$r^2 \sin 2\theta = a^2.$$

24. Perpendiculars are drawn from the pole to the tangents of the curve $r^2 \cos 2\theta = a^2$. Prove that the locus of the intersections is the curve

$$r^2 = a^2 \cos 2\theta.$$

25. Draw the curve $r^2 = a^2 \cos 2\theta$, and prove that it cuts itself at right angles.

26. Find the ratio in which the line joining xy, $x'y'$ is divided by the curve $xy = c^2$, and hence prove that the equation to the two tangents from $x'y'$ is

$$(xy' + x'y - 2c^2)^2 = 4(xy - c^2)(x'y' - c^2).$$

27. A chord PQ meets the asymptotes in $P'Q'$. Prove that $PP' = QQ'$.

28. A line meets the curves $xy = c^2$, $xy = c'^2$ in P, Q; P', Q'. Prove that $PP' = QQ'$.

29. A right-angled triangle is inscribed in a rectangular hyperbola. Prove that the hypothenuse is parallel to the normal at the right angle.

30. If a rectangular hyperbola circumscribe a triangle, it passes through the orthocentre.

31. If TP, TQ be tangents from T and PQ meet a directrix in Z, then TZ subtends a right angle at the focus.

32. Find the foci of the curve

$$xy = c^2 \;\ldots\ldots\; (\omega).$$

33. Given the foci, to what form does the hyperbola tend as the eccentricity diminishes?

34. Given the points A, A', to what form does the curve tend as the eccentricity diminishes?

35. Given the centre and directrices, to what form does the curve tend as the eccentricity diminishes?

36. Prove geometrically the diametral properties of a rectilinear hyperbola, and give a geometrical construction for drawing the polar of a given point.

Where are the foci and directrices of a rectilinear hyperbola?

37. Tangents are drawn to a conic section from any point of a fixed circle through the foci. Prove that the line bisecting the angle between the tangents passes through a fixed point.

38. From the equation

$$y = mx - \frac{(a^2 + b^2)\, m^2}{\sqrt{(a^2 - b^2 m^2)}}$$

deduce the equation to the normal to the parabola.

39. The semilatus-rectum of any conic section is a harmonic mean between the segments of any focal chord.

40. The segments of any focal chord of a conic section subtend equal angles at the foot of the directrix.

41. If one focus of a rectangular hyperbola be the point (a, b) and the corresponding directrix the line $\frac{x}{a} + \frac{y}{b} - 1$, then at the other focus

$$\frac{x}{a\,(3a^2 - b^2)} = \frac{y}{b\,(3b^2 - a^2)} = \frac{1}{a^2 + b^2}\,.$$

42. A circle intercepts chords of given length from two given straight lines. Find the locus of its centre.

43. Given two conjugate hyperbolas, find the locus of the pole, with respect to one, of any tangent to the other.

44. Point out among the examples given on the ellipse (Chapter IX.) those which can be adapted to the hyperbola.

CHAPTER XI.

GENERAL PROPOSITIONS.

167. We have seen that the equation to every conic section is of the second degree, and therefore included in the general form of the equation of the second degree

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0 \;\ldots\ldots\ldots\; (1).$$

We shall now prove that every curve whose equation is of the form (1) is a conic section, whatever be the inclination of the co-ordinate axes to which (1) refers.

The peculiar arrangement of terms and coefficients in (1) is adopted in order to ensure symmetry when a third symbol z is introduced, so as to make the equation take the homogeneous form

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0 \;\ldots\ldots\ldots\; (2).$$

The quantity z is variable only in form, being unity: it may be looked upon as a third co-ordinate of the point xy. Thus the points xyz, $x'y'z'$ are the points xy, $x'y'$.

We shall denote the left-hand sides of (1) and (2) by $\phi\,(x, y)$ and $f(x, y, z)$.

When the origin is transferred to $x'y'z'$ without changing the direction of the axes, (1) and (2) become

$$\phi\,(x + x', y + y') = 0, \qquad f(x + x', y + y', z + z') = 0.$$

Now $f(x + x', y + y', z + z') = f(x, y, z) + f(x', y', z') +$ an expression which may be written in two ways, viz.

$$2x\,(ax' + c'y' + b'z') + 2y\,(c'x' + by' + a'z') + 2z\,(b'x' + a'y' + cz') \;\; (\alpha),$$
$$2x'\,(ax + c'y + b') + 2y'\,(c'x + by + a'z) + 2z'\,(b'x + a'y + cz) \;\; (\beta).$$

If in either of the expressions (α), (β) we interchange x and x', y and y', and z and z', we obtain the other expression.

Of course we obtain $\phi(x+x', y+y')$ from $f(x+x', y+y', z+z')$ by making z and z' equal to 1.

The terms of highest dimensions in (1) are not altered by merely transferring the origin. This is the case with all rational and integral algebraic equations.

168. *The general equation* $\phi(x, y)=0$ *represents a conic section.*

I. Let the terms of highest dimensions $ax^2+by^2+2c'xy$ form a perfect square $(fx+gy)^2$.

Then $$\phi(xy)=(fx+gy)^2+2b'x+2c'y+c.$$

If the lines $fx+gy$, $2b'x+2c'y+c$ be not parallel, let them be made respectively the axes of x and y. Then $fx+gy$, varying as the distance of the point xy from the line $fx+gy$, becomes an expression varying as y, and $2b'x+2c'y+c$ becomes in like manner an expression varying as x. Thus the equation is reduced to

$$y^2=\lambda x,$$

λ being some constant, and this represents a parabola referred to a diameter and tangent.

Thus $$(fx+gy)^2+2b'x+2a'y+c=0 \;\ldots\ldots\ldots\ldots(1)$$

represents a parabola to which the line $2b'x+2a'y+c$ is a tangent, $fx+gy$ being the corresponding diameter.

If the lines $fx+gy$, $2b'x+2a'y+c$ be parallel, (1) represents two parallel lines. These are a parabola whose axis is a parallel midway between the lines, and whose focus and directrix are at infinity. (See Chap. VIII. Ex. 32.)

Thus, if the terms of highest dimensions in $\phi(x, y)$ form a perfect square, that is, if $ab-c'^2=0$, the locus of the equation $\phi(xy)=0$ is a parabola, and *by equating the perfect square to zero we get the direction of the axis.*

II. Let $ab-c'^2$ be not zero. If we transfer the origin to $x'y'$ we get an equation in which the terms of one dimension are

$$(ax'+c'y'+b')\,x+(c'x'+by'+a')\,y.$$

The coefficients of x and y vanish if

$$x' = \frac{a'c' - bb'}{ab - c'^2} \text{ and } y' = \frac{b'c' - aa'}{ab - c'^2}.$$

Let $\bar{x}$, $\bar{y}$ denote these values of x' and y'. Then the point $\bar{x}\bar{y}$ is at a finite distance, and by transferring the origin to $\bar{x}\bar{y}$ the equation becomes

$$ax^2 + by^2 + 2c'xy + \phi(\bar{x}\bar{y}) = 0,$$

or, if k denote $-\phi(\bar{x}\bar{y})$,

$$ax^2 + by^2 + 2c'xy = k \text{ .... } (2).$$

If $ab - c'^2$ be negative, the left-hand side resolves into two linear factors $Ax + By$, $Cx + Dy$.

Let the lines $Ax + By$, $Cx + Dy$, which intersect, be made axes, then (2) becomes $xy =$ a constant, which represents a hyperbola referred to its asymptotes.

Thus (2) represents a hyperbola of which the lines

$$ax^2 + by^2 + 2c'xy = 0 \text{ .... } (3)$$

are the asymptotes, and, if $ab - c'^2$ be negative, or, if the terms of highest dimensions break up into factors, then $\phi(xy) = 0$ represents a hyperbola whose asymptotes are parallel to the lines to which the factors correspond. In other words, *the directions of points at infinity are found by equating to zero the terms of highest dimensions.*

If ω be the inclination of the co-ordinate axes, the lines

$$c'(x^2 - y^2) + (b - a)xy = \cos\omega\,(ax^2 - by^2) \text{ .... } (4)$$

bisect the angles between the lines (3). (Cf. Art. 58 or 190.) They are therefore the axes of figure of the hyperbola (2). By making them co-ordinate axes (2) can be reduced to the form

$$Ex^2 + Fy^2 = k \text{ .... } (5),$$

E and F having opposite signs. For (5) is the form of equation to a hyperbola referred to its axes of figure.

Again, if $ab - c'^2$ be positive, the lines (3) are imaginary, but the lines (4) are still real and may be made co-ordinate axes. (For the condition of reality is

$$(b - a)^2 + 4c'^2 + 4ab\cos^2\omega > 4(a + b)c'\cos\omega,$$

or $\quad (b - a)^2(\cos^2\omega + \sin^2\omega) + 4c'^2 + 4ab\cos^2\omega > 4(a + b)c'\cos^2\omega,$

or $\quad \{(b + a)\cos\omega - 2c'\}^2 + (b - a)^2\sin^2\omega > 0).$

Then $ax^2 + by^2 + 2c'xy$ will still become $Ex^2 + Fy^2$, because, though the *values* of E and F be altered by altering those of a, b, c', yet their *forms* remain the same. The equation (2) becomes therefore

$$Ex^2 + Fy^2 = k.$$

Here E and F have the same sign, for otherwise the curve would be a hyperbola and the lines (3) therefore real.

Therefore the curve is an ellipse of which the squares of the semi-axes are $\frac{k}{E}$, $\frac{k}{F}$. If the sign of k be not the same as that of E and F, the ellipse is imaginary.

The lines (3) are imaginary asymptotes of the ellipse (2). The lines (4) are this ellipse's axes of figure. If $E = F$, and $\omega = \frac{\pi}{2}$, the ellipse becomes a circle. A circle is an ellipse with coincident foci and infinitely small eccentricity.

If $k = 0$, and $ab - c'^2$ be negative, (2) represents two intersecting straight lines. Two such lines are a rectilinear self-asymptotic hyperbola.

If $k = 0$ and $ab - c'^2$ be positive, (2) represents two imaginary lines which, as the form $Ex^2 + Fy^2 = 0$ assures us, are a point-ellipse.

Thus all curves represented by the equation $\phi(xy) = 0$ are conic sections.

169. *To find the centre of the conic* $\phi(\mathrm{xy}) = 0$.

Let $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre. Then $\phi(x + \bar{x}, y + \bar{y})$ has no linear terms. Hence $\bar{x}$, $\bar{y}$ are the values of x, y in the equations

$$ax + c'y + b' = 0, \quad c'x + by + a' = 0.$$

These values are

$$\frac{a'c' - bb'}{ab - c'^2}, \quad \frac{b'c' - aa'}{ab - c'^2}.$$

The centre of a parabola is generally at infinity. But if

$a'c' - bb'$ and $b'c' - aa'$ both vanish, the centre is indeterminate. In this case

$$abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2 = 0,$$

or the parabola is rectilinear. Its centre may be supposed anywhere in the axis.

170. When the axes are rectangular they are turned through an angle θ by writing $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y. When this is done, $ax^2 + 2c'xy + by^2$ is transformed to an expression of the same form in which the a, b, and c' are

$$a\cos^2\theta + b\sin^2\theta + 2c'\sin\theta\cos\theta,$$

$$a\sin^2\theta + b\cos^2\theta - 2c'\sin\theta\cos\theta,$$

and $$(b-a)\sin\theta\cos\theta + c'(\cos^2\theta - \sin^2\theta),$$

or $$\frac{a+b}{2} + \frac{a-b}{2}\cos 2\theta + c'\sin 2\theta,$$

$$\frac{a+b}{2} - \frac{a-b}{2}\cos 2\theta - c'\sin 2\theta,$$

and $$\frac{b-a}{2}\sin 2\theta + c'\cos 2\theta.$$

Thus *the values of* $\mathrm{a+b}$ *and* $\mathrm{ab-c'^2}$ *are not altered by turning the axes.* On this account $a+b$ and $ab - c'^2$ are called *invariants.*

Again, if we so choose θ that $\tan 2\theta = \dfrac{2c'}{a-b}$, we transform $ax^2 + by^2 + 2c'xy$ to the form $Ex^2 + Fy^2$. The innumerable values of θ differ by multiples of $\dfrac{\pi}{2}$.

Another invariant in the transformation of $\phi(xy)$ by turning the axes is $a'^2 + b'^2$, for $b'x + a'y$ becomes

$$(b'\cos\theta + a'\sin\theta)\,x + (a'\cos\theta - b'\sin\theta)\,y.$$

171. *To find the lengths of the axes of the conic*

$$\mathrm{ax^2 + by^2 + 2c'xy = k},$$

the co-ordinate axes being rectangular.

The given equation can be transformed to $\alpha x^2+\beta y^2=1$, α and β being the reciprocals of the squares of the semi-axes. (In the case of the hyperbola either α or β is negative.)

Thus

$$\frac{a}{k}x^2+\frac{b}{k}y^2+\frac{2c'}{k}xy \quad (1),$$

may be transformed to $\alpha x^2+\beta y^2$ (2) by turning.

The $a+b$ and $ab-c'^2$ of (1) are $\frac{a+b}{k}$ and $\frac{ab-c'^2}{k^2}$, and those of (2) are $\alpha+\beta$ and $\alpha\beta$.

Therefore $\alpha+\beta=\frac{a+b}{k}$ and $\alpha\beta=\frac{ab-c'^2}{k^2}$, and thus α, β are roots of the equation

$$\lambda^2-\frac{a+b}{k}\lambda+\frac{ab-c'^2}{k^2}=0.$$

The equation to the axes is (Art. 167)

$$c'(x^2-y^2)+(b-a)xy=0.$$

172. The asymptotes of $\phi(xy)=0$ are parallel to

$$ax^2+by^2+2c'xy=0 \text{..................} (1).$$

Therefore if θ be one of the angles between them,

$$\tan\theta=\pm\frac{2\sqrt{c'^2-ab}}{a+b}.$$

Let θ be that angle in which the curve lies; then, from Geometry, $\sec\frac{\theta}{2}=e$. Thus

$$\pm\frac{2\sqrt{c'^2-ab}}{a+b}=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{2\sqrt{e^2-1}}{2-e^2}.$$

It will be found that if μ denote $\frac{4c'^2+(a-b)^2}{c'^2-ab}$,

$$e^4-\mu e^2+\mu=0 \text{......................} (2).$$

Since $\mu^2 - 4\mu$, being $\dfrac{(a+b)^2}{(c'^2-ab)^2}\{4c'^2+(a-b)^2\}$ is positive, both values of e^2 are real. But if $ab > c'^2$, μ is negative and therefore only one value of e^2 is positive. Thus if $\phi(xy) = 0$ be an ellipse, we can find the eccentricity from (2). If $ab < c'^2$, that equation is not sufficient. The curve is a hyperbola, and the mere directions of the asymptotes are not sufficient to determine in which angle of the asymptotes the curve is situated.

The condition for a rectangular hyperbola is $a + b = 0$.

173. The curves obtained by varying k in

$$ax^2 + by^2 + 2c'xy = k$$

have the same centre and asymptotes. Those for which k is positive lie in differeut compartments from those for which k is negative. For the $ax^2 + by^2 + 2c'xy$ of any point on one curve has a different sign from the $ax^2 + by^2 + 2c'xy$ of any point on the other curve. That is, the curves are parted by the asymptotes (see Art. 177).

The asymptotes of $(Ax + By + C)(Dx + Ey + F) = G$ are the lines $Ax + By + C$, $Dx + Ey + F$, and the conjugate hyperbola is $(Ax + By + C)(Dx + Ey + F) = -G$, since the transformation which reduces one equation to $xy = h$ reduces the other to $xy = -h$.

Thus $ax^2 + by^2 + 2c'xy = \pm k$ are conjugate curves.

174. All the conics obtained by varying a', b' and c in $\phi(xy) = 0$ have their asymptotes parallel. Those obtained by varying only c are concentric and have the same asymptotes. Thus

$$\phi(xy) = \text{constant}$$

is concentric and co-asymptotic with $\phi(xy) = 0$, and only one such curve can be drawn through a given point $x'y'$, viz.

$$\phi(xy) = \phi(x'y').$$

Let $\bar{x}\,\bar{y}$ be the centre of $\phi(xy) = 0$. Then $\phi(xy) = \phi(\bar{x}\bar{y})$ (1) must be the asymptotes.

If in the expressions (α) or (β) of Art. 167, we write x, y, z for x', y', z', the result is $2f(xyz)$.

Thus $f(\bar{x}\,\bar{y}\,\bar{z}) = \bar{z}\,(b'\bar{x} + a'\bar{y} + c\bar{z})$,

and $\phi\,(\bar{x}\,\bar{y}) = b'\bar{x} + a'\bar{y} + c$

$$= \frac{b'(a'c' - bb') + a'(b'c' - aa')}{ab - c'^2} + c \text{ by Art. 169.}$$

Thus (1) may be written

$$ax^2 + by^2 + 2a'y + 2b'x + 2c'xy + \frac{aa'^2 + bb'^2 - 2a'b'c'}{ab - c'^2} = 0.$$

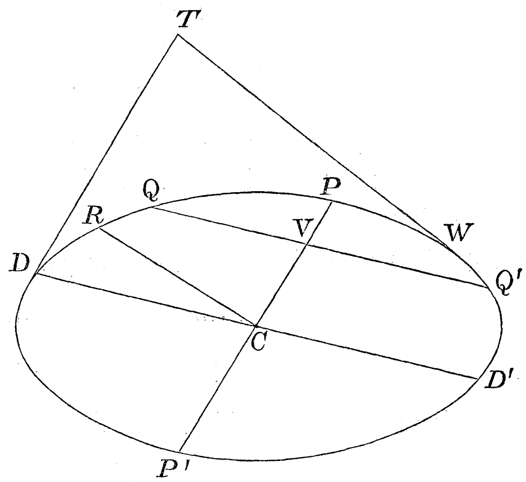
175. *If through any point there be drawn two lines in fixed directions to meet a conic section, the rectangles under the segments of the lines are in a constant ratio.*

(This Theorem includes Euclid III. 35, 36.)

Let any position of the two lines be made co-ordinate axes, and let

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0$$

be the equation to the curve. The rectangle of the intercepts



on the axis of x is, by the Theory of Quadratic Equations, $\frac{c}{a}$, and similarly $\frac{c}{b}$ is the rectangle of the intercepts on the axis of y. The rectangles are in the ratio $\frac{b}{a}$, which, as a and b do not change except with the *direction* of the axes, is invariable.

Ex. Let QVQ' be a double ordinate to the diameter PP' in an ellipse, and let DCD' be the diameter conjugate to PP'. Then

$$\frac{VQ.VQ'}{VP.VP'}=\frac{CD.CD'}{CP.CP'} \text{ or } \frac{QV^2}{PV.P'V}=\frac{CD^2}{CP^2}.$$

Again, let TD, TW be tangents parallel to the central radii CP, CR. Then

$$\frac{CP^2}{CR^2}=\frac{TD^2}{TR^2} \text{ or } \frac{CP}{CR}=\frac{TD}{TR}.$$

It follows also from this theorem, that if a pair of straight lines contain the four common points of a circle and any conic section, the straight lines are equally inclined to the axes of the curve, for the parallel central radii are equal.

By making two of the points move to coincidence with a third, we see that the common chord of the conic and the circle of curvature at any point makes the same angles with the axes as the tangent at the point where the circle of curvature is drawn.

176. The equation for determining the length r of a line drawn from a given point $x'y'$ in direction $[l, m]$ to meet the conic $\phi(xy)=0$ is

$$r^2(al^2+bm^2+2c'lm)+2r\{l(ax'+c'y'+b')+m(c'x'+by'+a')\}$$
$$+\phi(x'y')=0 \text{ ....(1).}$$

Thus the diameter conjugate to the direction (l, m) is the line

$$l(ax+c'y+b')+m(c'x+by+a')$$

which passes through the intersection of the lines

$$ax+c'y+b, \quad c'x+by+a',$$

that is, through the centre.

If the centre be origin, the equation to the diameter may be written in two ways:

$$l\,(ax + c'y) + m\,(c'x + by) = 0,$$

or
$$x\,(al + c'm) + y\,(c'l + bm) = 0.$$

The condition that the directions (lm), $(l'm')$ may be conjugate is

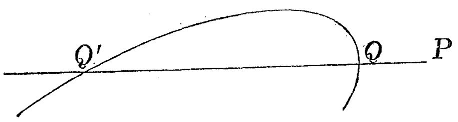
$$all' + bmm' + c'\,(lm' + l'm) = 0,$$

and is the same for all curves which have the same asymptotic directions.

177. The product of the values of r in equation (1) of Art. 176 is

$$\frac{\phi\,(x'y')}{al^2 + bm^2 + 2c'lm}.$$

Thus the $\phi\,(xy)$ of any point P varies as the rectangle of the segments of a line PQQ' drawn through P in a given direction to meet the curve.



The $\phi\,(xy)$ of a point changes sign as the point crosses the curve, for when the point is without the curve the segments are measured in the same direction, and their rectangle is positive, but when the point is within the curve, the segments are measured in opposite directions and their rectangle is negative.

If $al^2 + bm^2 + 2clm$ be positive, the $\phi\,(xy)$ of all points without the curve is positive.

A point is within or without the curve according as $\phi\,(xy)$ has or has not the same sign as $\phi\,(\bar{x}\,\bar{y})$.

Arts. 175 and 177 are particular cases of theorems which hold for all curves that are represented by rational and integral algebraic equations.

Similar Curves.

178. By magnifying or diminishing the radii vectores of any curve in a constant ratio we get another curve of the same shape, or *similar* to the given curve. Moreover the two curves are similarly placed: corresponding chords, for instance, are parallel. If the radii of the derived curve be backward productions of those of the given curve, the curves are still similar and have corresponding chords parallel, but differ in position by two right angles. The origin, or point whence the radii are drawn, is a centre of *direct* similitude in the first case, and of *inverse* similitude in the second.

Thus the curves

$$\frac{l}{r} = 1 + e \cos \theta \text{.................. (1)},$$

$$\frac{l'}{r} = 1 + e \cos \theta \text{.................. (2)},$$

are similar and similarly placed and their common focus is a centre of direct similitude. The curves

$$\frac{l}{r} = 1 + e \cos \theta,$$

$$\frac{l'}{r} = 1 + e \cos (\theta - \alpha)$$

are similar, but differ in position by an angle α. The curve

$$\frac{l'}{r} = 1 - e \cos \theta,$$

is similar to (1) and similarly placed, and the common focus is a centre of inverse similitude.

If two conics of the same eccentricity be applied together so as to have a common focus and the same direction of transverse axes, their equations can be exhibited in the forms (1), (2). Therefore the curves are similar, and conversely similar conics have the same eccentricity.

All parabolas are similar curves.

All similar and similarly placed conics must have virtually the same highest terms in their Cartesian equations, since their asymptotes are in the same directions.

But if $\phi_1(xy)$ denote

$$ax^2 + by^2 + c_1 + 2a_1'y + 2b_1'x + 2c'xy,$$

it is not necessarily true that the curves ϕ, ϕ_1 are similar. For their equations referred to their centres xy, x_1y_1 are

$$ax^2 + by^2 + 2c'xy = -\phi(\bar{x}\bar{y}),$$
$$ax^2 + by^2 + 2c'xy = -\phi_1(\bar{x}_1, \bar{y}_1),$$

and if $ab < c'^2$, it is not certain, unless the signs of $\phi(\bar{x}\bar{y})$, $\phi_1(\bar{x}, \bar{y})$ be known, in which angles of their asymptotes the curves lie.

The curves $\phi(xy) = 0$, $\phi_1(xy) = 0$ are similar except $\phi(\bar{x}\bar{y})$, $\phi(\bar{x}_1 \bar{y}_1)$ have opposite signs, and then their eccentricities e, e_1 are connected by the relation

$$\frac{1}{e^2} + \frac{1}{e_1^2} = 1.$$

Tangents, Poles and Polars.

179. Let $\mu : \mu'$ be the ratio in which the line joining the points xyz, $x'y'z'$ is cut by the conic $f(xyz) = 0$. The co-ordinates of the point of section are

$$\frac{\mu x' + \mu' x}{\mu + \mu'}, \quad \frac{\mu y' + \mu' y}{\mu + \mu'}, \quad \frac{\mu z' + \mu' z}{\mu + \mu'},$$

and these satisfy the equation to the curve. Therefore, since f is homogeneous,

$$f(\mu x' + \mu' x,\ \mu y' + \mu' y,\ \mu z' + \mu' z) = 0.$$

The expanded form of this is

$$\mu^2 U' + \mu'^2 U + \mu\mu' V = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

if U, U' denote $f(xyz)$, $f(x'y'z')$, and V denote either of the expressions (α) or (β) in Art. 167.

I. Let the line touch the curve. Then the two values of the ratio $\mu : \mu'$ are equal; the condition for which is

$$V^2 = 4UU' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If the point $x'y'z'$ be given, and x, y, z be 'current' co-ordinates, (2) is the equation to the two tangents from $x'y'z'$.

Ex. To find the equation to the two tangents from $x'y'$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Here $$f(xyz) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2,$$

and $$V = 2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - zz'\right).$$

Therefore the equation (2) is

$$\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - zz'\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2\right)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - z'^2\right),$$

or $$\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right).$$

II. Let the line touch the curve at $x'y'z'$. Then both values of $\mu : \mu'$ are infinite, so that both U' and V vanish. Thus

$$V = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is the equation to the tangent at $x'y'z'$.

Ex. The tangent at $x'y'$ to the curve $\phi(xy) = 0$ is

$$(ax' + c'y' + b')\,x + (c'x' + by' + a')\,y + b'x' + a'y' + c = 0 \ldots (4).$$

It can be shewn (as in Art. 86) that (4) is the equation to the chord of contact of two tangents from $x'y'$.

III. Let the line joining the two points be cut harmonically by the curve. Then the values of $\mu : \mu'$ are equal and of opposite sign: that is,

$$V = 0.$$

Thus, if $x'y'z'$ be given, $V = 0$ is the equation to the locus of the other point. The locus is therefore a straight line.

Hence, if through a given point a radius vector be drawn to a conic and the harmonic mean between its two values be measured upon it from the given point, the locus of the extremity of this new radius vector is a straight line.

This straight line is called the *polar* of the given point, and the given point is called the *pole* of the given line (see Art. 90).

Ex. The polar of the origin with respect to $\phi(xy) = 0$ is, since V in this case is

$$z'(b'x + a'y + c),$$

$$b'x + a'y + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The polar form of $\phi(xy) = 0$ is

$$r^2(al^2 + bm^2 + 2c'lm) + 2r(b'l + c'm) + c = 0,$$

and the harmonic mean between the values of r is

$$-\frac{c}{b'l + c'm}.$$

Thus, the equation to the locus of the extremity of the harmonic mean is

$$r(b'l + c'm) + c = 0,$$

which agrees with (5).

180. *To find the axis of the parabola*

$$(fx + gy)^2 + 2a'y + 2b'x + c = 0.$$

The polar of $x'y'$ is

$$x\{f(fx'+gy')+b'\}+y\{g(fx'+gy')+a'\}+\ldots=0\ldots(1),$$

and the *direction* of the axis is (Art. 168)

$$fx+gy=0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The axis is the locus of points whose polars are perpendicular to the axis. Thus if $x'y'$ lie on the axis, the lines (1), (2) are at right angles, or

$$f^2(fx'+gy')+fb'+g^2(fx+gy)+ga'=0.$$

The equation to the axis is therefore

$$(f^2+g^2)(fx+gy)+fb'+ga'=0.$$

181. *To find the latus rectum of the parabola*

$$(fx+gy)^2+2b'x+2a'y+c=0.$$

Let θ be the angle between the lines

$$fx+gy,\ 2b'x+2a'y+c,$$

so that
$$\sin\theta=\frac{gb'\sim fa'}{\sqrt{(f^2+g^2)(a'^2+b'^2)}},$$

when these lines are made axes of x and y,

$$\frac{fx+gy}{\sqrt{f\ \ +g^2}}$$

will become $y\sin\theta$, and

$$\frac{2b'x+2a'y+c}{2\sqrt{a'^2+b'^2}}$$

will become $-x\sin\theta$, so that the equation will be

$$y^2(f^2+g^2)\sin\theta=2x\sqrt{a'^2+b'^2}.$$

Now, if 4α be the latus rectum, we know that another form of this equation is

$$y^2=4\alpha\operatorname{cosec}^2\theta\,.\,x \ \text{(Art. 114)}.$$

Therefore
$$4\alpha\operatorname{cosec}^2\theta=\frac{2\sqrt{a'^2+b'^2}}{f^2+g^2}\operatorname{cosec}\theta,$$

and $$4\alpha = \frac{2\,(gb' \sim fa')}{(f^2+g^2)^{\frac{3}{2}}}.$$

182. *To find the director-circle of the conic* $\phi\,(\mathrm{xy}) = 0$.

The lines
$$V^2 - 4UU' = 0 \dots\dots\dots\dots\dots\dots (1)$$
are at right angles if the sum of the coefficients of x^2 and y^2 in (1) vanishes.

Now
$$\frac{V^2 - 4UU'}{4} = \{(ax' + c'y' + b')\,x + (c'x' + by' + a')\,y + \dots\}^2$$
$$- U'.\,(ax^2 + by^2 + 2c'xy + \dots).$$

And the sum of the coefficients of x^2 and y^2 vanishes if
$$(ax' + c'y' + b')^2 + (c'x' + by' + a')^2 = (a + b)\;U'.$$

The equation to the director-circle is therefore
$$(ax + c'y + b')^2 + (c'x + by + a')^2$$
$$= (a + b)\,(ax^2 + by^2 + c + 2a'y + 2b'x + 2c') \dots\dots (2).$$

In the case of the parabola, the terms of highest dimensions in (2) disappear, and (2) becomes the equation to the directrix, viz.
$$2b'\,(ax + c'y) + 2a'\,(c'x + by) + a'^2 + b'^2 = (a + b)\,(c + 2a'y + 2b'x),$$
or $$2\,(a'c' - bb')\,x + 2\,(b'c' - aa')\,y + a'^2 + b'^2 - (a + b)\,c = 0 \dots (3).$$

Again, for determining the pole of a line $Ax + By + C$ we have the equations
$$\frac{ax + c'y + b'}{A} = \frac{c'x + by + a'}{B} = \frac{b'x + a'y + c}{C} \dots\dots (4).$$

We can therefore find the pole of the line (3), that is, the focus of the parabola.

183. *All chords of a conic section which subtend a right angle at a fixed point of the curve, intersect in the normal at that point.* (Compare Euclid, III. 31.)

The fixed point being origin, let the equation to the conic be

$$ax^2 + by^2 + 2a'y + 2b'x + 2c'xy = 0.$$

The finite intercepts on the axes are

$$-\frac{2b'}{a}, \quad -\frac{2a'}{b}.$$

The chord which joins the extremities of these intercepts is

$$\frac{x}{-\frac{2b'}{a}} + \frac{y}{-\frac{2a'}{b}} = 1 \text{ .... } (1).$$

The tangent at the origin is (Art. 100)

$$a'y + b'x = 0.$$

The normal is therefore

$$\frac{y}{a'} = \frac{x}{b'} \text{ .... } (2).$$

The lines (1) and (2) meet in the point

$$x = \frac{-2b'}{a+b}, \quad y = \frac{-2a'}{a+b}.$$

The distance of this point from the origin is

$$\frac{2\sqrt{a'^2 + b'^2}}{a+b},$$

which does not change when the axes are turned (Art. 170). The chords therefore all pass through a fixed point in the normal.

184. *Through five points, no four of which are in a straight line, one conic section and one only can be drawn.*

Let the axes be so chosen that two of the points are in one axis, and two of the rest in the other axis, and let these four points be

$$(\alpha, 0), \ (\alpha', 0), \ (0, \beta), \ (0, \beta').$$

Assume for the equation to the conic

$$Ax^2 + By^2 + Cxy + Dx + Ey + 1 = 0 \text{.......... } (1).$$

Then, by the theory of quadratic equations,

$$A = \frac{1}{\alpha\alpha'}, \quad -D = \frac{1}{\alpha} + \frac{1}{\alpha'}, \quad B = \frac{1}{\beta\beta'}, \quad -E = \frac{1}{\beta} + \frac{1}{\beta'}.$$

Thus (1) takes the form

$$\frac{x^2}{\alpha\alpha'} + \frac{y^2}{\beta\beta'} + Cxy - \left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)x - \left(\frac{1}{\beta} + \frac{1}{\beta'}\right)y + 1 = 0 \ldots\ldots (2).$$

Let $x_1\, y_1$ be the fifth point. Then by writing x_1, y_1 for x, y in (2) we obtain a simple equation for determining C. Thus one and only one conic can be drawn through the five points. If the point $x_1\, y_1$ be in either axis, C is infinite, and therefore (1) must be $xy = 0$, representing a rectilinear hyperbola. If four of the points lie in a straight line, there can be drawn through the five points an infinite number of hyperbolas, all rectilinear.

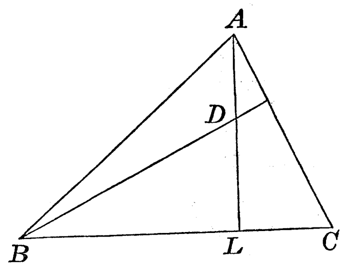
Cor. The equation to a conic which passes through four given points can always be thrown into the form (2).

185. If $S = 0$ (1) and $S' = 0$ (2) be the equations to two conics, then $S + \lambda S' = 0$ (3) denotes a conic passing through their four points of intersection, and by properly choosing λ may be made to represent *any* such conic, for λ can be so chosen as to make (3) satisfied by the co-ordinates of any fifth point.

186. Hence every conic section which passes through the intersections of two rectangular hyperbolas is also a rectangular hyperbola. For if the $a + b$ of S and the $a + b$ of S' both vanish, so does the $a + b$ of $S + \lambda S'$.

187. In the triangle ABC let AL be perpendicular to BC, and let any rectangular hyperbola which passes through A, B, C meet AL again in D.

The lines AL, BC are a rectangular hyperbola intersecting the curve in A, B, C, D. Therefore the lines BD, AC, which are a rectilinear hyperbola, are also a rectangular hyperbola. Therefore BD is at right angles to AC. Thus



If a rectangular hyperbola circumscribe a triangle, it passes also through the intersection of perpendiculars.

188. The reader of the Differential Calculus can well apply the notation of that subject to Plane Co-ordinate Geometry.

To begin with Art. 167. By a well-known expansion

$$f(x+x',\ y+y',\ z+z') = f(xyz) + x'\frac{df}{dx} + y'\frac{df}{dy} + z'\frac{df}{dz}$$

$$+ x'^2\frac{d^2f}{dx^2} + \ldots\ldots\ldots + 2y'z'\frac{d^2f}{dy\,dz} + \ldots\ldots$$

+ terms which involve higher differential coefficients and therefore vanish,

$$= f(xyz) + x'\frac{df}{dx} + y'\frac{df}{dy} + z'\frac{df}{dz} + f(x'y'z').$$

The terms of one dimension in x, y, z are

$$x'\frac{df}{dx} + y'\frac{df}{dy} + z'\frac{df}{dz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or, if f stand for $f(x'y'z')$,

$$x\frac{df}{dx'} + y\frac{df}{dy'} + z\frac{df}{dz'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Thus the V of Art. 179 stands for either of the expressions (1), (2), and the equation to the polar of any point is found by equating either of these expressions to zero.

The equations for determining the pole of a given line

$$Ax + By + C \dots\dots\dots\dots\dots\dots (3),$$

as found by comparing (3) with (2), are

$$\frac{\frac{df}{dx}}{A} = \frac{\frac{df}{dy}}{B} = \frac{\frac{df}{dz}}{C}.$$

189. Again, the equations for determining the centre of the conic $\phi(xy) = 0$ are simply

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0,$$

(each of which represents, therefore, a diameter).

The $\phi(xy)$'s of two points in the same diameter and equidistant from the centre are equal (Art. 177). At the centre $\phi(xy)$ is a maximum or minimum.

Also, since $\quad 2f(xyz) = x\frac{df}{dx} + y\frac{df}{dy} + z\frac{df}{dz},$

$$f(\bar{x}\bar{y}\bar{z}) = \frac{1}{2}\frac{df}{d\bar{z}} \text{ (compare Art. 174).}$$

190. The normal at any point $x'y'$ of $\phi(xy) = 0$ is the line

$$\frac{x - x'}{\frac{d\phi}{dx'}} = \frac{y - y'}{\frac{d\phi}{dy'}}.$$

Let $\psi(xy)$ denote $ax^2 + 2c'xy + by^2$, so that

$$\psi(xy) = k \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

is the general equation to a conic referred to its centre. Then the normal at $x'y'$ is

$$\frac{x - x'}{\frac{d\psi}{dx'}} = \frac{y - y'}{\frac{d\psi}{dy'}}.$$

If this line pass through the origin of co-ordinates, that is, if the normal be one of the axes of figure,

$$\frac{x'}{\frac{d\psi}{dx'}} = \frac{y'}{\frac{d\psi}{dy'}},$$

or, $x'y'$ lies on one of the lines

$$x\frac{d\psi}{dy} - y\frac{d\psi}{dx} = 0 \qquad \text{(2)};$$

which, be it observed, intersect at the origin, and are therefore the axes of figure.

The lines $$\psi(xy) = 0 \qquad \text{(3)},$$
are the asymptotes of the curve $\psi(xy) = k$, and, since the axes of the curve bisect the angles between the asymptotes, *the equation* (2) *represents the bisectors of the angles between the lines* (3).

191. The equation to the axis of the parabola

$$(ax + by + c)^2 + dx + ey + f = 0 \qquad \text{(1)},$$

is, if χ denote the left-hand side of (1),

$$a\frac{d\chi}{dx} + b\frac{d\chi}{dy} = 0.$$

The equation to the director-circle of $\phi(xy)$ is

$$\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 = 4(a+b)\,\phi(xy).$$

The equation to the diameter conjugate to chords in direction (lm) is

$$l\frac{d\phi}{dx} + m\frac{d\phi}{dy} = 0,$$

and, if the centre be origin, this may be also written

$$x\frac{d\phi}{dl} + y\frac{d\phi}{dm} = 0 \quad \{\phi \text{ standing for } \phi(lm)\}.$$

192. We conclude this Chapter by tracing three curves from their equations.

I. The locus of

$$(x + y - 2)^2 = y - 3x \qquad \text{(1)}$$

is a parabola touching the line $y - 3x$ at the point where that line meets the line $x + y - 2$ (which is a diameter) and lying on

the positive side of the line $y-3x$, since the $y-3x$ of every point in the curve is equal to a perfect square. The equations for determining the intercepts on the axes are

$$x^2-x+4=0, \quad y^2-5y+4=0.$$

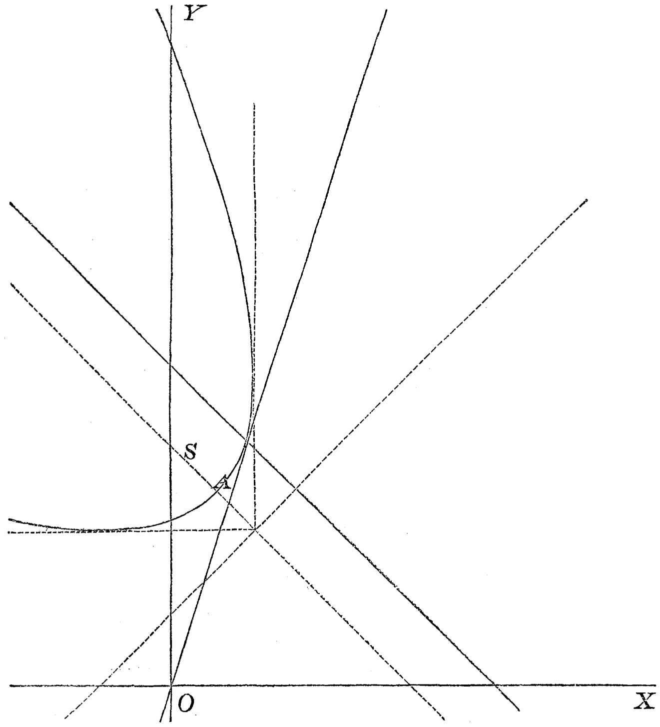
Thus the curve does not meet the axis of x in real points, but meets the axis of y at the points (0, 1), (0, 4). From these observations the curve can be roughly drawn, whatever be the inclination of the axes.

But, to be more precise, if the axes be rectangular, the axis of figure is (Art. 180) the line

$$x+y-2=-\frac{1}{2} \quad \text{.........................(2)},$$

and the vertex, as we find by combining (1) and (2), is the point

$$x=\frac{5}{16}, \quad y=1\tfrac{3}{16}.$$



The latus rectum $\left(\text{Art. } 181\text{; since the } f, g, a', b' \text{ are } 1, 1, -\frac{1}{2}, -\frac{5}{2}\right)$ is $\sqrt{2}$, and the focus, which we find by adding to the coordinates of A the projections of SA on the axes, is the point

$$x = \frac{1}{16}, \quad y = 1\tfrac{7}{16}.$$

In like manner, the directrix is a line through the point

$$x = \frac{5}{16} + \frac{1}{4}, \quad y = 1\tfrac{3}{16} - \frac{1}{4},$$

at right angles to the line $x + y$: that is, the directrix is the line

$$x - y + \frac{3}{8} = 0.$$

Transferring the origin to the points (0, 1), (0, 4), we find the terms of lowest dimensions to be in one case $x - 3y$, and in the other $7x + 3y$. Thus we know the directions of the curve at the points where it cuts the axis of y.

Again, to find the points where the tangent is parallel to the axis of x, we must find the value of y for which the values of x in (1) are equal. That value of y is $\frac{15}{16}$, and the corresponding value of x is $-\frac{7}{16}$. Thus the line $y = \frac{15}{16}$ is a tangent, and in like manner the line $x = \frac{9}{32}$ is a tangent, intersecting the former in the foot of the directrix.

II. The equation

$$x^2 - xy + 2y^2 = 8x + 3y + 5 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

represents an ellipse whose centre is at the point (5, 2).

With this point as origin the equation is

$$x^2 - xy + 2y^2 = 28 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for $\phi(\bar{x}\bar{y}) = b'\bar{x} + a'\bar{y} + c$ (Art. 174).

The co-ordinates being rectangular, the axes of figure are

$$x^2 - 2xy - y^2 = 0 \text{ (Art. 170 or 190),}$$

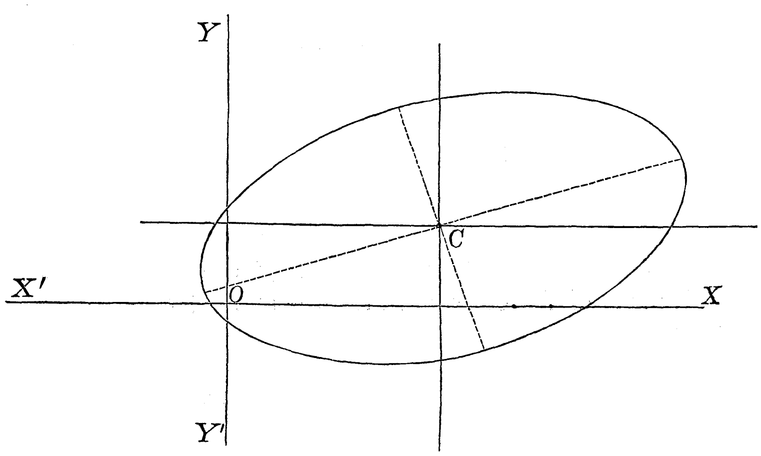
$$\text{or } y = (\pm\sqrt{2} - 1)\,x.$$

The polar form of (2) is

$$\frac{28}{r^2} = \cos^2\theta - \cos\theta\sin\theta + 2\sin^2\theta.$$

Let $\tan\theta = m$, so that

$$\sin\theta = \frac{m}{\sqrt{1+m^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+m^2}}.$$



Then $$\frac{28}{r^2} = \frac{1 - m + 2m^2}{1+m^2},$$

and, if $$m = \pm\sqrt{2} - 1, \quad m^2 = 3 \mp 2\sqrt{2}$$

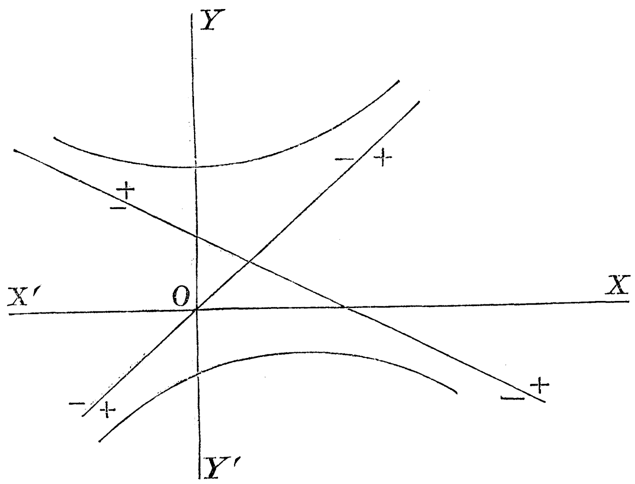
and $$\frac{r^2}{28} = \frac{4 \mp 2\sqrt{2}}{8 \mp 5\sqrt{2}} = \frac{2 \pm 4\sqrt{2}}{14}.$$

Thus the squares of the semi-axes are $8(3 \pm \sqrt{2})$, and the longer axis is in direction of the line $y = (\sqrt{2} - 1)\,x$.

III. The equation

$$(x-y)(x+2y-3)+7=0$$

represents a hyperbola of which the asymptotes are the lines $x-y$ and $x+2y-3$.



The curve lies on the positive side of one line and on the negative side of the other, or in the $+-$ and $-+$ compartments of the two lines.

The equation for the intercepts on the axis of x is

$$x^2-3x+7=0,$$

the roots of which are imaginary. Thus the curve does not cut the axis of x in real points. The equation for intercepts on the axis of y is $2y^2-3y-7=0$, the roots of which are $\frac{3 \pm \sqrt{65}}{4}$.

The curve can now be roughly drawn: but if greater accuracy be required the reciprocals of the squares of the semi-axes may be found from the equation

$$\lambda^2-\frac{1}{7}\lambda-\frac{9}{196}=0 \text{ (Art. 171).}$$

Examples on Chapter XI.

1. Find an equation to the parabola $(x+y-3)^2=2x-y$, referred to the lines $x+y-3$, $2x-y$ as axes.

2. Find the equation to the parabola which passes through the point (h, k), touches the line $x=y$ at the origin, and has the axis of y for a diameter............ (ω).

3. Find the equation to the curve $x^2-y^2-3xy+x=0$, referred to its centre. What is the eccentricity of this curve?

4. Determine to what classes of conic sections the following curves belong:

$$(1)\quad \sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$$

$$(2)\quad x^2-3xy+y^2=9x-7y.$$

$$(3)\quad x^2+xy+3=0.$$

5. All conics obtained by varying m in the equation

$$x^2+mxy+y^2=1$$

have their axes coincident .... (ω).

6. Find the angle through which the axes must be turned in order that

$$x^2-5xy+6y^2$$

may take the form

$$Ex^2+Fy^2,$$

and reduce the equation

$$x^2-5xy+6y^2-4x-2y=5$$

to the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

7. Find the lengths of the axes of the curves

$$x^2 - xy + y^2 = 1, \qquad x^2 - xy - y^2 = 1.$$

8. Prove that the length of an equi-conjugate radius vector in the ellipse

$$ax^2 + by^2 + 2c'xy = k \text{ is } \sqrt{\frac{k(a+b)}{2(ab-c'^2)}},$$

and hence prove that the equation to the equi-conjugate diameters is

$$(a+b)(ax^2 + by^2 + 2c'xy) = 2(ab - c'^2)(x^2 + y^2).$$

9. Find the equation to the common diameters of the conic

$$ax^2 + by^2 + 2c'xy = 1,$$

and the circle $\qquad \lambda(x^2 + y^2) = 1,$

and deduce the condition that the curves may touch.

Hence find the lengths of the semi-axes of the conic and the equation to the axes.

10. The conics

$$Ax^2 + By^2 + 2C'xy + \ldots = 0,$$

$$ax^2 + by^2 + 2c'xy + \ldots = 0,$$

have their axes parallel if

$$\frac{A-B}{C'} = \frac{a-b}{c'}.$$

11. Find the eccentricities of

$$5x^2 + 3y^2 - 2xy + x - 3y = 0,$$

$$(x-y)x = 2,$$

$$x^2 + xy + y^2 = a^2,$$

and find the equation to a rectangular hyperbola which touches the axis of x at the point $(1, 0)$, and the line $x = y$ at the point $(1, 1)$.

12. Determine λ and λ' so that

$$(Ax + By + \lambda)(A'x + B'y + \lambda')$$

may differ by a constant from

$$(Ax + By)(A'x + B'y) + Cx + Dy,$$

and hence find the asymptotes of

$$(Ax + By)(A'x + B'y) + Cx + Dy + E = 0.$$

13. Find the asymptotes of

$$(x - y)(x + 3y) + 4x - y = 6.$$

14. What is represented by the equation

$$(Ax + By + C)^2 + (A'x + B'y + C')^2 = D^2?$$

15. Determine μ so that $\phi(xy) + \mu$ may break up into two linear factors, and hence find the asymptotes of the conic

$$\phi(xy) = 0.$$

16. If TP, TQ be two tangents to a parabola whose focus is S,

$$TP^2 : TQ^2 :: SP : SQ.$$

17. If a conic touch three straight lines BC, CA, AB in L, M, N, then the lines AL, BM, CN meet in a point, and the intersections of the three chords of contact with the corresponding sides of the triangle ABC lie in a straight line.

18. If a conic touch all the sides of a rectilineal figure, the products of the alternate segments of the sides are equal.

19. Find the equation to the common chord of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the circle of curvature at the point $x'y'$, and find the length of this chord.

What must be the eccentricity of the ellipse if the common chord of the ellipse and circle of curvature at an extremity of a latus rectum pass through the further focus?

20. Apply Art. 176 to find the equation to the axes of the conic $$ax^2 + by^2 + 2c'xy = k.$$

21. Is the point $(1, -2)$ within or without the curve
$$y^2 = 6x + 2y + 1\,?$$

22. Is the point $(1, 1)$ within or without the curve
$$(5x - 4y)^2 - (3x + 2y - 1)^2 = 5\,?$$

23. Find the ratio of focal chords parallel to the co-ordinate axes in the conic
$$x^2 + 3xy - 2y^2 + 5x = 3.$$

24. Find a centre of similitude of two squares which are in similar positions, and hence inscribe a square in a given triangle.

25. From the focus of a conic a line is drawn inclined at a constant angle to the tangent. Find the locus of the point where the line meets the tangent.

26. The curves obtained by varying c in the equation
$$F\left(\frac{x}{c}, \frac{y}{c}\right) = 0,$$
are similar and similarly situated......... (ω).

27. Find the equation to the axis of the parabola
$$(fx + gy)^2 + 2a'y + 2b'x + c = 0,$$
the co-ordinates being oblique.

28. Find the director-circle of the conic
$$ax^2 + by^2 + 2c'xy = k......(\omega).$$

29. A parabola slides between two given straight lines which are at right angles. Determine the loci of the focus and vertex.

30. Trace the curves
$$y = x - x^2, \qquad x = y - y^2,$$
and find the angles at which they intersect.

31. Find the axis of the parabola

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \ldots\ldots\ldots (\omega).$$

32. Prove by Art. 183 that if a right-angled triangle be inscribed in a rectangular hyperbola, the hypothenuse is parallel to the normal at the right angle.

33. A right-angled triangle is inscribed in the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Prove that the locus of the point where the hypothenuse meets the normal at the right angle is the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.$$

What is this locus in the case of the parabola?

34. Find the polar of the origin with regard to the conic

$$\frac{x^2}{\alpha\alpha'} + \frac{y^2}{\beta\beta'} + Cxy - x\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right) - y\left(\frac{1}{\beta} + \frac{1}{\beta'}\right) + 1 = 0,$$

and prove that if a conic circumscribe a quadrilateral, then the line joining any two of the vertices of the quadrilateral is the polar of the third vertex. (Art. 78.)

35. Prove generally for all conic sections the last theorem in Art. 101, and prove that the centre of the conic $\phi(xy)$ is the pole of the line at infinity.

36. With a given centre one conic, and in general one only, can be described about a given triangle. Examine the cases in which the given centre is in a side of the triangle.

37. If a conic pass through four given points, the locus of its centre is a conic.

38. Assuming that the locus of the centre of a conic passing through four given points A, B, C, D is a conic, prove *a priori* that on this locus lie the intersections of AD, BC; BD, CA; CD, AB; and the middle points of BC, CA, AB, AD, BD, CD.

39. The circle which bisects the sides of a triangle passes through the feet of the perpendiculars from the angular points on the sides, and bisects the distances of the angular points from the orthocentre.

40. If a rectangular hyperbola circumscribe a triangle, the locus of its centre is the 'nine-point circle.'

41. How many parabolas can be drawn through four given points? How many rectangular hyperbolas?

42. With a given focus how many conics can be described about a given triangle?

43. Apply Art. 187 to the problem in Ex. 32.

44. Find the equation to the mid-parallel of the lines

$$Ax + By + C, \qquad Ax + By + C' \quad \text{(Art. 180)}.$$

45. The lines $L - \mu M$, $L + \mu M$ are conjugate diameters of the curve $LM = k$.

46. Find the points in the conic $\phi(xy) = 0$, when the tangents are parallel to the co-ordinate axes.

47. Trace the curves

(1) $x^2 + xy = x + 5$. (2) $(x + y - 5)^2 = y - 2x + 8$.

(3) $(x - y)^2 + (x + 3y)^2 = 16$.

(4) $(x - y + 2)(x - 3y + 3) + 5 = 0$;

and find the equation to a conic which has the same centre and asymptotes as (4) and passes through the centre of (1).

CHAPTER XII.

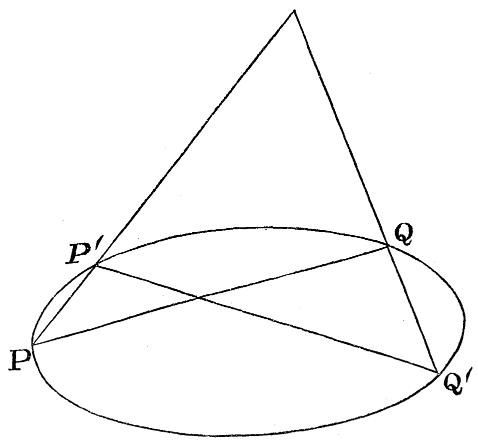
ABRIDGED NOTATION.

193. Let $L=0$, $M=0$ be the equations to two chords PQ, $P'Q'$ of the conic

$$S=0 \text{..............................} (1).$$

Then the equation $LM=0$ represents a rectilinear hyperbola, and (by Art. 185) the equation to every conic passing through the points P, Q, P', Q', is of the form

$$S+\lambda\, LM=0 \text{........................} (2).$$



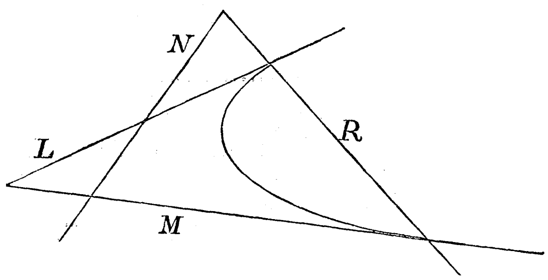
Let the point P', Q' move up to and coincide with the points P, Q respectively. Then the lines PP', QQ', which were common chords of the conics (1), (2), become common tangents at P, Q, and we see that

$$S+\lambda\, L^2=0 \text{.....................} (3),$$

represents a conic having double contact with (1) 'along the

line' L. Since (3) is a simple equation as far as λ is concerned, only one conic can be drawn through a given point so as to have double contact with a given conic along a given line.

194. Let $L=0$, $M=0$, $N=0$, $R=0$, be the equations to four straight lines. Then (Art. 193), $LM+\lambda NR=0$ is the general equation to a conic passing through the four points in which the lines LM meet the lines NR: and $LM+\lambda R^2$ is the general equation to a conic touching the lines LM and having R for the chord of contact.



Ex.
$$(fx+gy)^2+2a'y+2b'x+c=0$$
touches the line
$$2a'y+2b'x+c$$
and the line at infinity, and has $fx+gy$ for the chord of contact.

195. Thus the equation to a conic referred to two tangents as axes must be of the form
$$xy+\lambda\,(Ax+By+C)^2=0,$$
$Ax+By+C$ being the chord of contact.

A more convenient form is
$$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=\mu xy \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In this case a and b are the lengths of the tangents.

Ex. The conditions that (1) may represent a circle are
$$a^2=b^2, \text{ and } \frac{2}{ab}-\mu=\frac{2\cos\omega}{a^2},$$
ω being the inclination of the axes.

Thus the equation to a circle referred to two tangents of length a inclined at an angle ω is

$$(x+y-a)^2 = 4xy \sin^2 \frac{\omega}{2}. \quad \text{(Compare Ch. VI. Ex. 32.)}$$

This equation, translated into trilinear co-ordinates by calling the lines x, y, $x+y-a$ respectively α, β, γ, is $\alpha\beta = \gamma^2$. (The triangle of reference is, in this case, isosceles.)

Again, the condition for a parabola is

$$\frac{1}{a^2b^2} = \left(\frac{1}{ab} - \frac{\mu}{2}\right)^2, \text{ or } \mu = \frac{4}{ab}.$$

When this condition is satisfied the equation (1) can be reduced to

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

196. Let e be the eccentricity of a conic having the origin for focus and the line $\alpha = 0$ for directrix. Then the equation to the curve is

$$x^2 + y^2 = e^2 \alpha^2,$$

or
$$(e\alpha - y)(e\alpha + y) = x^2,$$

or
$$(e\alpha - x)(e\alpha + x) = y^2.$$

Thus the lines $e\alpha - y$, $e\alpha + y$ are tangents at the points where the axis of x meets the curve, and the lines $e\alpha - x$, $e\alpha + x$ are tangents at the points where the axis of y meets the curve.

197. We might define a focus of a conic as a point such that if it be made origin the equation to the curve can be put into the form

$$x^2 + y^2 = e^2\gamma^2 \text{ ....}(1),$$

or as a point whose distance from any point of the curve can be expressed as a linear function of the co-ordinates of the point. But, to speak of imaginary things as if they were real, a focus is

'a point-circle having double contact with the conic.' For the curve

$$x^2+y^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

has double contact with the curve (1) along the line γ.

Again, a conic meets its asymptotes in the line at infinity, and if the direction of the asymptotes be given (that is, if the terms of highest dimensions in the conic's equation be given), the conic meets the line at infinity in two given points. Thus all similar and similarly placed ellipses have two imaginary common points at infinity, and, in particular, all circles pass through 'the two circular points at infinity,' which are the points at infinity on the lines

$$(x-a)^2+(y-b)^2=0.$$

Call these points θ, θ', and let S be any point whatever. Then all circles centred at S have the same asymptotes, viz. the lines $S\theta$, $S\theta'$, and these lines are themselves one of the concentric system, viz. the point-circle S.

Let S, S' be the foci of any conic. Then the lines $S\theta$, $S\theta'$, $S'\theta$, $S'\theta'$, all touch the curve, and if from θ, θ' there be drawn tangents so as to form a circumscribing quadrilateral, two of the angular points of this quadrilateral are the foci S, S'. What are the other two angular points? Imaginary foci. For if they be σ, σ', then $\sigma\theta$, $\sigma\theta'$ touch the curve, and therefore by making σ origin, the equation to the curve can be put into the form $x^2+y^2=e^2\gamma^2$. The directrix γ is in this case imaginary. Similarly σ' is a point-circle having double contact with the curve.

198. *To find the foci of the conic* ϕ (xy) $=0$.

The tangents from the point $x'y'$ are represented by

$$V^2-4UU'=0 \ldots\ldots\ldots\ldots(1) \quad \text{(Art. 179)}.$$

If the point $x'y'$ be without the curve these lines are a rectilinear hyperbola: if on the curve, a rectilinear parabola, if within the curve, a point-ellipse which (like all point-ellipses) is rectilinear, and, if the point be a focus, the lines are a point-circle.

Thus the conditions under which (1) represents a circle make the point $x'y'$ a focus. These conditions are, that the coefficients of x^2 and y^2 be equal and that the coefficient of xy vanish.

Therefore since (1) is

$$\{x(ax'+cy'+b')+y(c'x'+by'+a')+\dots\}^2=U'(ax^2+by^2+2c'xy+\dots),$$

the equations for determining the foci are

$$\left.\begin{array}{c}(ax+c'y+b')^2-(c'x+by+a')^2=(a-b)\ U\\ (ax+c'y+b')\,(c'x+by+a')=c'\ U\end{array}\right\}\dots\dots\dots(2).*$$

Ex. Let $U=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$.

Then for a, b, c, a', b', c' we must write

$$\frac{1}{a^2},\ \frac{1}{b^2},\ -1,\ 0,\ 0,\ 0,$$

so that the equations (2) become

$$\frac{x^2}{a^4}-\frac{y^2}{b^4}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),$$

and $$\frac{xy}{a^2b^2}=0,$$

or $$x^2-y^2=a^2-b^2 \text{ and } xy=0\ \dots\dots\dots\dots\dots\ (3).$$

Thus the foci are in the axes of figure at the points

$$\pm\sqrt{a^2-b^2},\ 0,\ \text{and}\ \pm\sqrt{b^2-a^2},\ 0.$$

Of these points two S, S' are real, and two σ, σ' imaginary, and in any conic the lines SS', $\sigma\sigma'$ are real and at right angles. The equations (2) represent rectangular hyperbolas. In fact every conic which passes through the four foci of a conic is a rectangular hyperbola, since it passes through the intersections of two rectangular hyperbolas, viz. the axes of figure and the hyperbola whose vertices are the four foci.

* Another form of (2) is

$$\frac{\overline{\frac{d\phi}{dx}}\big|^2-\overline{\frac{d\phi}{dy}}\big|^2}{a-b}=\frac{\frac{d\phi}{dx}\cdot\frac{d\phi}{dy}}{c'}=4\phi.$$

In the case of the parabola three of the foci are at infinity, and from the equations (2) are deduced two linear equations determining the fourth focus.

199. *Confocal Conics.*

The equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad (1),$$

can, by giving a proper value to λ, be made to represent any conic which is confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (2).$$

(Observe that a^2 and b^2 are not necessarily *positive* quantities: thus (2) may represent a hyperbola.)

How many confocals to (2) can be drawn through a given point $x'y'$?

The equation for determining λ is

$$(\lambda+a^2)(\lambda+b^2)-(\lambda+a^2)y^2-(\lambda+b^2)x^2=0 \qquad (3).$$

Suppose a^2 algebraically greater than b^2, so that $-\infty$, $-a^2$, $-b^2$, $+\infty$, are in order of magnitude.

The left-hand side of (3), when these four quantities are successively written for λ, takes the signs $+$, $+$, $-$, $+$, and therefore changes sign (and so vanishes) for some value of λ between $-a^2$ and $-b^2$, and for some value greater than $-b^2$. Thus (3) has real roots, and one of these roots makes $\lambda+a^2$, $\lambda+b^2$ have different signs, while the other gives them the same sign. Hence through any given point there can be drawn two confocals to a given conic, a confocal ellipse and a confocal hyperbola. Of course if the given point be on the given conic, one of these confocals is the conic itself.

Again, confocals cut at right angles. For if $x'y'$ be a common point of (1) and (2), then by subtraction

$$\frac{x'^2}{(a^2+\lambda)\,a^2}+\frac{y'^2}{(b^2+\lambda)\,b^2}=0 \qquad (4),$$

or, the tangent lines

$$\frac{xx'}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}-1, \quad \frac{xx'}{a^2}+\frac{yy'}{b^2}-1$$

are at right angles. In order that $x'y'$ may be real, we see from (4) that $(a^2+\lambda)\,a^2$ and $(b^2+\lambda)\,b^2$ must have opposite signs: that is, the confocals must be of different species, one an ellipse, one a hyperbola.

The equation $y^2=4a\,(x+a)$ can by varying a be made to represent any parabola having one real focus at the origin, and the other at infinity in direction of the axis of x. Thus two confocals can be drawn through a given point; and these intersect at right angles, for in the equation $y'^2=4a\,(x'+a)$ the product of the values of $\frac{2a}{y'}$ is -1.

200. Let $\alpha=0$, $\beta=0$, $\gamma=0$ be the equations to three lines forming a triangle of reference, and let these equations be so written that the triangle is on the positive side of each line. Then at any point in the plane of operations

$$a\alpha+b\beta+c\gamma=2\Delta.$$

The equation to any conic can be thrown into the form

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0 \;\ldots\ldots\ldots\; (1),$$

in which u, v, w, u', v', w' are constants.

For let the sides CA, CB be made axes of x and y. Then the new equation to the curve must be of the form

$$Lx^2+My^2+N+2L'y+2M'x+2N'xy=0.$$

But the new x and y are (see Art. 70) the old $\frac{\alpha}{\sin C}, \frac{\beta}{\sin C}$.

Therefore the old equation was

$$L\alpha^2+M\beta^2+N\sin^2 C+2L'\beta\sin C+2M'\alpha\sin C+2N'\alpha\beta=0;$$

which, made homogeneous, is

$$L\alpha^2 + M\beta^2 + N\sin^2 C\left(\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right)^2$$
$$+ (L'\beta \sin C + M'\alpha \sin C)\left(\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right) + 2N'\alpha\beta = 0\,;$$

and this homogeneous equation is of the form (1).

If the equations to three lines not meeting in a point be $L = 0$, $M = 0$, $N = 0$, the equation to any conic can be thrown into the form

$${}_1L^2 + v_1M^2 + w_1N^2 + 2u_1'MN + 2v_1'NL + 2w_1'LM = 0 \,\ldots\ldots(2),$$

in which u_1, v_1, w_1, u_1', v_1, w_1' are constants.

For if $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the other forms of the linear equations, the equation to the curve takes the form (1).

Now α, β, γ vary as L, M, N. Therefore the equation (1) can be written in the form (2).

201. By the method of Art. 179 (see Chap. v. Ex. 11) it can be shewn that the equation

$$\alpha\,(u\alpha' + w'\beta' + v'\gamma') + \beta\,(w'\alpha' + v\beta' + u'\gamma')$$
$$+ \gamma\,(v'\alpha' + u'\beta' + w\gamma') = 0 \,\ldots\ldots\ldots(1),$$

or

$$\alpha'\,(u\alpha + w'\beta + v'\gamma) + \beta\,(w'\alpha + v\beta + u'\gamma)$$
$$+ \gamma\,(v'\alpha + u'\beta + w\gamma) = 0 \,\ldots\ldots\ldots(2),$$

represents the tangent at the point $\alpha'\beta'\gamma'$ or the polar of the point $\alpha'\beta'\gamma'$, and that the equation to the two tangents from $\alpha'\beta'\gamma'$ is

$$V^2 = 4UU' \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

it being now understood that U, U', V stand for new expressions obtained by changing

$$x,\ y,\ z,\ x',\ y',\ z',\ a,\ b,\ c,\ a',\ b',\ c'$$

to

$$\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma',\ u,\ v,\ w,\ u',\ v',\ w'$$

respectively.

If $f(\alpha\beta\gamma)$ denote the left side of the curve's equation, (1), (2) may be written

$$\alpha\frac{df}{d\alpha'}+\beta\frac{df}{d\beta'}+\gamma\frac{df}{d\gamma'}=0, \qquad \alpha'\frac{df}{d\alpha}+\beta'\frac{df}{d\beta}+\gamma'\frac{df}{d\gamma}=0.$$

202. To find the pole of a given line

$$l\alpha+m\beta+n\gamma=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

Let $\alpha'\beta'\gamma'$ be the pole; then equation (1) is identical with

$$(u\alpha'+w'\beta'+v'\gamma')\,\alpha+(w'\alpha'+v\beta'+u'\gamma')\,\beta$$
$$+(v'\alpha'+u'\beta'+w\gamma')\,\gamma=0.$$

Hence the equations for determining the co-ordinates of the pole are

$$\frac{u\alpha+w'\beta+v'\gamma}{l}=\frac{w'\alpha+v\beta+u'\gamma}{m}=\frac{v'\alpha+u'\beta+w\gamma}{n},$$

or
$$\frac{1}{l}\cdot\frac{df}{d\alpha}=\frac{1}{m}\cdot\frac{df}{d\beta}=\frac{1}{n}\cdot\frac{df}{d\gamma}.$$

203. *To find the centre of the conic.*

The centre is the pole of the line at infinity, that is, of the line
$$a\alpha+b\beta+c\gamma=0.$$

Therefore the equations for determining the co-ordinates of the centre are

$$\frac{u\alpha+w'\beta+v'\gamma}{a}=\frac{w'\alpha+v\beta+u'\gamma}{b}=\frac{v'\alpha+u'\beta+w\gamma}{c},$$

or
$$\frac{1}{a}\cdot\frac{df}{d\alpha}=\frac{1}{b}\cdot\frac{df}{d\beta}=\frac{1}{c}\cdot\frac{df}{d\gamma}.$$

204. If $f(\alpha\beta\gamma)=0$ be the equation in a rational and integral form to a conic of which $\bar\alpha\bar\beta\bar\gamma$ is the centre, then

$$f(\alpha\beta\gamma)=f(\bar\alpha\bar\beta\bar\gamma)$$

is the equation to the asymptotes. This result may be deduced from Art. 174, or may be established thus:

The general equation to a conic having double contact with $f(\alpha\beta\gamma)$ along the line $l\alpha + m\beta + n\gamma$ is (Art. 194),

$$f(\alpha\beta\gamma) + \lambda\,(l\alpha + m\beta + n\gamma)^2 = 0.$$

If the line be the line at infinity, the equation becomes

$$f(\alpha\beta\gamma) = \text{a constant}.$$

Thus only one conic can be drawn through a given point $\alpha'\beta'\gamma'$ having double contact at infinity with the given conic $f(\alpha\beta\gamma) = 0$, namely the conic

$$f(\alpha\beta\gamma) = f(\alpha'\beta'\gamma').$$

The equation to the one such conic through the centre, that is, to the asymptotes, is

$$f(\alpha\beta\gamma) = f(\bar{\alpha}\bar{\beta}\bar{\gamma}).$$

205. The general equation to a conic circumscribing the triangle of reference is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \text{........................(1).}$$

For if the point A (that is the point when $\beta = 0$ and $\gamma = 0$) lie on the curve

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \text{......(2),}$$

then $u = 0$.

Similarly if B and C lie on the curve, $v = 0$ and $w = 0$, and thus (2) is reduced to a form equivalent to (1).

Writing (1) in the form $l\beta\gamma + \alpha\,(m\gamma + n\beta) = 0$, we see that the points where the line $m\gamma + n\beta$ meets the conic lie in the two straight lines $\beta\gamma = 0$. Now the $m\gamma + n\beta$ is drawn through the intersection of the lines $\beta\gamma$. Therefore those points coincide, or the line $m\gamma + n\beta$ is the tangent at the point A. Similarly the lines $n\alpha + l\gamma$, $l\beta + m\alpha$ are tangents at B, C.

206. If the circumscribing conic be a circle, then by Euclid III. 32 the three tangents must be the lines

$$\gamma \sin B + \beta \sin C, \quad \alpha \sin C + \gamma \sin A, \quad \beta \sin A + \alpha \sin B;$$

hence

$$\frac{l}{\sin A} = \frac{m}{\sin B} = \frac{n}{\sin C}, \text{ or } \frac{l}{a} = \frac{m}{b} = \frac{n}{c},$$

and the equation to the circumscribing circle is

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0.$$

Let $S = 0$ be the Cartesian equation to any particular circle. Then a form of the general equation to a circle is

$$S + Ax + By + C = 0.$$

Thus a form of the general equation to a circle in Trilinears is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + L_1\alpha + M_1\beta + N_1\gamma = 0,$$

or

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(L\alpha + M\beta + N\gamma) = 0.$$

207. The line $L\alpha + M\beta + N\gamma = 0$ touches the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

if, on combining the equations and eliminating γ, the resulting equation give equal values of the ratio $\alpha : \beta$; that is, if

$$\sqrt{\frac{l}{L}} + \sqrt{\frac{m}{M}} + \sqrt{\frac{n}{N}} = 0.$$

Thus the conic is a parabola if

$$\sqrt{\frac{l}{a}} + \sqrt{\frac{m}{b}} + \sqrt{\frac{n}{c}} = 0,$$

for all parabolas touch the line at infinity.

208. *To find the general equation to a conic touching the three lines of reference.*

In $f(\alpha\beta\gamma) = 0$ put $\alpha = 0$. There results

$$v\beta^2 + w\gamma^2 + 2u'\beta\gamma = 0,$$

which is the equation to the lines joining the point A to the points where BC meets the curve. If these lines coincide, that

is, if $u'^2 = vw$, BC touches the curve. Hence if the conic touch BC, CA, AB,

$$u'^2 = vw, \quad v'^2 = wu, \quad w'^2 = uv,$$

and $f(\alpha\beta\gamma) = 0$ reduces to

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2\sqrt{vw}\,.\,\beta\gamma + 2\sqrt{wu}\,.\,\gamma\alpha + 2\sqrt{uv}\,.\,\alpha\beta = 0.$$

Let $$u = l^2, \quad v = m^2, \quad w = n^2;$$

then we may write the equation in either of the forms

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0.$$

209. Thus the equation to the circle inscribed in the triangle of reference must be of the form

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0 \quad \text{..................} \quad (1)$$

At A', the point of contact with BC, $\alpha = 0$, and therefore

$$\sqrt{m\beta} + \sqrt{n\gamma} = 0, \text{ or } m\beta = n\gamma;$$

that is, $$mA'C \sin C = nA'B \sin B.$$

But $$A'C = \frac{\Delta}{s} \cot \frac{C}{2},$$

and $$A'B = \frac{\Delta}{s} \cot \frac{B}{2}.$$

Thus $$\frac{m}{n} = \frac{\cos^2 \frac{B}{2}}{\cos^2 \frac{C}{2}},$$

and $$\frac{l}{\cos^2 \frac{A}{2}} = \frac{m}{\cos^2 \frac{B}{2}} = \frac{n}{\cos^2 \frac{C}{2}}.$$

The equation to the inscribed circle is therefore

$$\cos \frac{A}{2} \sqrt{\alpha} + \cos \frac{B}{2} \sqrt{\beta} + \cos \frac{C}{2} \sqrt{\gamma} = 0.$$

Similarly the equation to the escribed circle in the angle A is

$$\cos\frac{A}{2}\sqrt{-\alpha}+\cos\frac{B}{2}\sqrt{\beta}+\cos\frac{C}{2}\sqrt{\gamma}=0.$$

210. Since any straight line can be represented by equations of the form

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=r,$$

it follows that the equation for determining the length of a line drawn from the point $\alpha'\beta'\gamma'$ 'in direction lmn' to meet the curve is

$$f(\alpha'+lr,\ \beta'+mr,\ \gamma'+nr)=0,$$

or

$$f(\alpha'\beta'\gamma')+r\{l(u\alpha'+w'\beta'+v'\gamma')+m(w'\alpha'+v\beta'+u'\gamma')$$
$$+n(v'\alpha'+u'\beta'+w\gamma')\}+r^2.f(lmn)=0 \ldots\ldots\ldots\ldots (1)$$

Thus the equation to the diameter conjugate to chords in direction lmn is

$$l(u\alpha+w'\beta+v'\gamma)+m(w'\alpha+v\beta+u'\gamma)$$
$$+n(v'\alpha+u'\beta+w\gamma)=0 \ldots\ldots\ldots\ldots (2)$$

or $$l\frac{df}{d\alpha}+m\frac{df}{d\beta}+n\frac{df}{d\gamma}=0,\ \text{or}\ \alpha\frac{df}{dl}+\beta\frac{df}{dm}+\gamma\frac{df}{dn}=0.$$

211. If the line be drawn parallel to the side BC, the values of l, m, n may be called 0, $\sin C$, $-\sin B$.

Suppose this line drawn from the centre; then by geometry or by Art. 203 the coefficient of r vanishes, and the equation for determining r is

$$f(\bar{\alpha}\bar{\beta}\bar{\gamma})+r^2 f(0,\ \sin C,\ -\sin B)=0.$$

Thus $$r^2=-\frac{f(\bar{\alpha}\bar{\beta}\bar{\gamma})}{v\sin^2 C+w\sin^2 B-2u'\sin B\sin C}.$$

Thus if r_a, r_b, r_c be the lengths of central radii parallel to the sides of the triangle of reference,

$$\frac{1}{r_a^2}:\frac{1}{r_b^2}:\frac{1}{r_c^2}::vc^2+wb^2-2u'bc:wa^2+uc^2-2v'ca$$
$$:ub^2+va^2-2w'ab.$$

COR. The conditions for a circle are

$$vc^2 + wb^2 - 2u'bc = wa^2 + uc^2 - 2v'ca = ub^2 + va^2 - 2w'ab.$$

212. Let A, B, C, D be four points on a conic (see figure to Art. 78), and let the three pairs of lines which contain them all meet respectively in L, M, N.

Then if MN meet CD, AB in F, G, the ranges $DFCL, AGBL$ are harmonic, for the lines MA, MN, MB, ML form a harmonic pencil (Art. 78). Therefore (Art. 179) the points F, G lie on the polar of L, or, MN is the polar of L.

Similarly LN is the polar of M, and therefore (as in Art. 101; cf. Chap. XI. Ex. 35) LM is the polar of N.

Thus of the three points L, M, N each is the pole of the line joining the other two. The three points are a *conjugate triad* with respect to the conic, and are a conjugate triad with respect to *all* conics passing through A, B, C, D.

As any two conics have four common points, real or imaginary, they have a conjugate triad.

213. *To determine the form of the equation to the conic when the vertices of the triangle of reference are a conjugate triad.*

The equation to the polar of any point $\alpha'\beta'\gamma'$ with respect to $f(\alpha\beta\gamma)$ is

$$\alpha'(u\alpha + w'\beta + v'\gamma) + \beta'(w'\alpha + v\beta + u'\gamma) + \gamma'(v'\alpha + u'\beta + w\gamma) = 0.$$

Make $\beta' = 0$ and $\gamma' = 0$, this becomes

$$u\alpha + w'\beta + v'\gamma = 0.$$

If this line coincide with the line α, $v' = 0$ and $w' = 0$.

Similarly if B be the pole of CA, $w' = 0$ and $u' = 0$.

Thus if the points A, B, C be a conjugate triad, the equation to the curve must be of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0.$$

The triangle of reference is said to be *self-conjugate*, being its own copolar (Chap. VII. Ex. 8).

214. *To find the condition that the curve* f $(\alpha\beta\gamma)$ *may be a rectangular hyperbola.*

The coefficients of x^2 and y^2 in the Cartesian equation must be equal and of opposite sign. The condition will be found to be, since the cosines of the *angles* $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ are the cosines of A, B, C,

$$u+v+w+2u'\cos A+2v'\cos B+2w'\cos C=0.$$

In the case of the curve $u\alpha^2+v\beta^2+w\gamma^2$ the condition is

$$u+v+w=0.$$

In the case of the curve

$$u'\beta\gamma+v'\gamma\alpha+w'\alpha\beta=0$$

the condition is

$$u'\cos A+v'\cos B+w'\cos C=0\,;$$

or, in geometrical language, the curve must pass through the orthocentre.

215. In any conic if SY, $S'Y'$ be the perpendiculars from the foci on a tangent, $SY \,.\, S'Y'=BC^2$. Therefore if a conic touch the lines

$$\alpha=0,\ \beta=0,\ \gamma=0,\ \delta=0\ldots\ldots,$$

and $\qquad \alpha,\ \beta,\ \gamma,\ \delta,\ldots\ \alpha',\ \beta',\ \gamma',\ \delta',\ldots$

be the distances of S, S' from these lines,

$$\alpha\alpha'=\beta\beta'=\gamma\gamma'=\delta\delta'=\ldots$$

Thus, if the locus of S be expressed homogeneously with reference to the lines α, β, γ, δ,... the locus of S' will be found by writing

$$\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma}\ldots \text{ for } \alpha,\ \beta,\ \gamma\ldots.$$

For example, if a parabola touch the three sides α, β, γ of a triangle of reference, since one focus is at infinity, or lies on the line

$$a\alpha+b\beta+c\gamma=0,$$

the other focus must lie on the curve

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=0,$$

that is, on the circle which circumscribes the triangle. (Art. 206.)

Examples on Chapter XII.

1. Determine λ so that the equation

$$x^2+y^2-c^2=\lambda\,(xx'+yy'-c^2)^2$$

may represent the two tangents from $x'y'$ to the circle $x^2+y^2-c^2$.

2. Determine λ so that $U=\lambda V^2$ (Art. 179) may represent the two tangents from $x'y'$ to the conic U.

Find the asymptotes of the conic $ax^2+2bxy+cy^2=f\ldots(\omega)$.

3. If the three pairs of sides of a triangle be asymptotes of three conics, then the three finitely-distant chords of intersection of the conics meet in a point.

4. Prove geometrically that if a parabola touch the three sides of a triangle its focus lies on the circumscribing circle, and hence prove that the focus of $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ is at the point determined by

$$ax=by,\quad x^2+y^2+2xy\cos\omega=\frac{ax+by}{2},$$

ω being the inclination of the axes.

5. Prove that any point on the conic $LM=R^2$ can be defined by the equations $\mu^2L=\mu R=M$, μ being an arbitrary constant, and find the equation to the chord joining the points μ, μ'. Deduce the equation to the tangent at the point μ.

6. TP, TQ are tangents to a conic and the bisector of the angle PTQ meets PQ in R. Prove that the segments of any chord through R subtend equal angles at T.

7. Find the equation to the conjugate axis of the curve

$$(x-1)^2 + (y+2)^2 = 5(x-3y)^2.$$

8. Determine the foci of the parabolas

$$y^2 = 4ax, \quad y^2 = 4(ax+by).$$

9. How many confocals can be to a given conic so as to touch a given line?

10. To any point xy on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

there corresponds a point $x'y'$ upon the confocal

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

such that

$$\frac{x}{a} = \frac{x'}{a'}, \quad \frac{y}{b} = \frac{y'}{b'};$$

and if the points P', Q' on one conic correspond to P, Q on the other, then $PQ' = P'Q$.

11. Form equations for determining the centres of the conics

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0, \qquad \sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0, \qquad \beta\gamma = k\alpha^2,$$

and deduce in each case the condition for a parabola.

12. If $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$ be a parabola, then the line

$$\frac{\beta}{na+lc} = \frac{\gamma}{lb+ma}$$

is a diameter (Ex. 11). Hence find the focus and prove that the directrix passes through the orthocentre of the triangle of reference.

13. Prove both by Art. 205 and Art. 208 that if a conic touch BC, CA, AB in A', B', C' the lines AA', BB', CC' meet in a point and the respective intersections $B'C'$, $C'A'$, $A'B'$ with BC, CA, AB lie in a straight line.

14. Prove that the equation to a circle bisecting the sides of the triangle of reference is

$$\frac{a}{\frac{\Delta}{a}-\alpha}+\frac{b}{\frac{\Delta}{b}-\beta}+\frac{c}{\frac{\Delta}{c}-\gamma}=0 \quad \text{(Art. 205).}$$

15. A conic touches the three sides of a triangle and has one focus in a given straight line. Prove that the other focus lies on a conic described about the triangle.

16. The semi-axes of an ellipse are a, b and the distances of its foci and centre from the lines α, β inclined at an angle ω are α, β, α', β', $\bar{\alpha}$, $\bar{\beta}$. Prove that

$$\alpha+\alpha'=2\bar{\alpha}, \quad \beta+\beta'=2\bar{\beta}, \quad \alpha\alpha'=\beta\beta'=b^2,$$

$$(\alpha-\alpha')^2+(\beta-\beta')^2+2(\alpha-\alpha')(\beta-\beta')\cos\omega=4\sin^2\omega\,(a^2-b^2).$$

17. An ellipse of semi-axes a, b slides between two straight lines inclined at an angle ω. Prove that the locus of the real foci referred to the two lines is

$$\left(x-\frac{b^2}{x\sin^2\omega}\right)^2+\left(y-\frac{b^2}{y\sin^2\omega}\right)^2$$
$$+2\left(x-\frac{b^2}{x\sin^2\omega}\right)\left(y-\frac{b^2}{y\sin^2\omega}\right)\cos\omega=4(a^2-b^2),$$

and find the locus of the centre.

18. A quadrilateral is inscribed in a conic and tangents are drawn at the angular points so as to form a circumscribing quadrilateral. Prove that the two quadrilaterals have a common intersection of diagonals.

CHAPTER XIII.

MISCELLANEOUS THEOREMS AND METHODS.

216. PASCAL'S THEOREM. *If a hexagon be inscribed in a conic, the three intersections of opposite sides will lie in a straight line.*

Take any six points on a conic section, and in any order whatever, call them 1, 2, 3, 4, 5, 6: then join them by six straight lines 12, 23, 34, 45, 56, 61, so as to form a hexagon of which the pairs of opposite angular points are 1, 4; 2, 5; 3, 6, and the pairs of opposite sides 12, 45; 23, 56; 34, 61. Then the three intersections of these pairs lie in a straight lie. By naming the six given points in different orders we have 60 hexagons which have the given points for angular points; and not more than one of these hexagons can be free from re-entrant angles. Pascal's Theorem is true for all.

Let the equations to 12, 23, 34, 45, 56, 61 be respectively $u=0$, $v=0$, $w=0$, $u'=0$, $v'=0$, $w'=0$, and let $z=0$ be the equation to the diagonal 14. Then the given conic circumscribes each of the quadrilaterals 1234, 3456, and therefore its equation can (Art. 194) be written in either of the forms

$$uw+\lambda vz=0 \text{.........................} (1),$$

$$u'w'+\lambda'v'z=0 \text{.........................} (2),$$

λ, λ' being constants.

Hence the expression $uw+\lambda vz$ is identically equal to

$$k\,(u'w'+\lambda'v'z),$$

k being some constant. Therefore $uw-ku'w'$ is identical with

$k\lambda'v'z - \lambda vz$, or with $(k\lambda'v' - \lambda v)z$, and therefore breaks up into two linear factors; and therefore the equations

$$uw - ku'w' = 0 \dots\dots\dots\dots(3),$$

$$(k\lambda'v' - \lambda v)\, z = 0 \dots\dots\dots\dots(4),$$

represent the same pair of straight lines, namely the line 14 and a line through the point where v, v' meet (or 'the point vv''). These two lines, as we see from (3), contain the points uu', ww', which, not lying on the line 14, must therefore lie on the line through vv'. Thus the points uu', vv', ww' lie in a straight line.

Ex. Let the points 2, 4, 6 coincide respectively with 1, 3, 5. Then the lines 12, 34, 56 become the tangents at 1, 3, 5, and the lines 23, 45, 61 become 13, 35, 51, and the theorem becomes the second theorem of Chap. XII. Ex. 13.

217. Pascal's Theorem may be proved for the circle by means of the following lemma from Trigonometry (see Chap. V. Exs. 16, 18, 23).

If L, M, N *be points in the sides* BC, CA, AB *of a triangle either such that* AL, BM, CN *meet in a point, or such that* LMN *is a straight line; then*

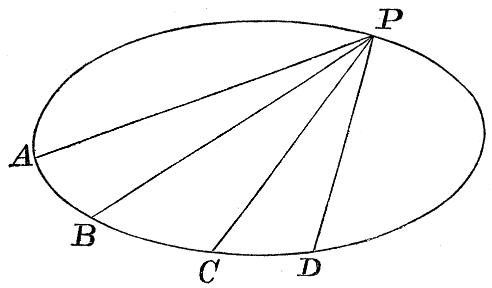
$$\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1 \dots (\alpha).$$

Conversely: Let L, M, N *be taken in the sides of the triangle so that this equation is satisfied; then if* L, M, N *lie one or all in the sides unproduced,* AL, BM, CN *meet in a point, but if* L, M, N *lie one or all in the sides produced,* LMN *is a straight line.*

For the alternate sides 12, 34, 56 of the hexagon inscribed in a circle form a triangle which has each of the remaining sides for a 'transversal.' Whence we get three equations like (α). Multiplying these together and making use of Euclid III. 35, 36, we get another equation like (α), proving that the three intersections 12, 45; 23, 56; 34, 61 lie in a transversal of the same triangle.

We shall presently shew how this theorem may be geometrically extended from the circle to the other conic sections.

218. *If four fixed points be taken on a conic the anharmonic ratio of the pencil formed by joining them to any fifth point on the conic is constant.* (See Chap. VII. Ex. 12.)



Let A, B, C, D be the four fixed points, and let the equations to AB, BC, CD, DA be $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$. Then the equation to the conic is (Art. 194) of the form $\alpha\gamma=k.\beta\delta$ (1).

Now the α of P is the altitude of the triangle having P for vertex and the chord AB for base, and therefore

$$=\frac{PA\,.\,PB\sin APB}{AB};$$

and similarly we can express the β, γ, δ of P. Hence (1) asserts that

$$\frac{\sin APB}{\sin CPB}\div\frac{\sin APD}{\sin CPD}=k,$$

or that the anharmonic ratio of PA, PB, PC, PD is constant.

The symbol $\{P.ABCD\}$ is used to denote the anharmonic ratio of the pencil PA, PB, PC, PD and $\{abcd\}$ to denote the anharmonic ratio of four points a, b, c, d lying on a straight line (or of the *range abcd*). Let A, B, C, D be as above and let $abcd$ be a transversal to the pencil. Then $\{P.ABCD\}$ is constant and $=\{abcd\}=\dfrac{ab}{cb}\div\dfrac{ad}{cd}$.

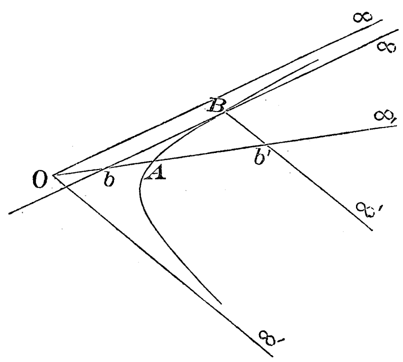
219. We take one example of this theorem from *Salmon's Conic Sections.*

If P be made to coincide with one of the given points, as A, the chord PA becomes the tangent at A, and the anharmonic ratio of the pencil is denoted by $\{A.ABCD\}$.

Now let ∞, ∞' denote the points at infinity on the conic, so that if O be the centre, $O\infty$, $O\infty'$ are the asymptotes. Take ∞, ∞' for the points C and D, and make P coincide first with ∞ and next with ∞'. Thus

$$\{\infty \,.\, AB \infty \infty'\} = \{\infty' \,.\, AB \infty \infty'\}.$$

The lines ∞A, ∞B are parallels to the asymptote $O\infty$: the line $\infty\infty$ is this asymptote, and the line $\infty\infty'$ is altogether at infinity.



Taking the line OA for the transversal, we have for a, b, c, d four points on the line OA, viz. A, b (where OA meets ∞B), $O, \infty_{,}$, and for $\frac{ab}{cb} \div \frac{ad}{bd}$ we have $\frac{Ab}{Ob} \div \frac{A\infty_{,}}{O\infty_{,}}$, or $\left(\text{since } \frac{A\infty_{,}}{O\infty_{,}} \text{ is a ratio of equality}\right) \frac{Ab}{Ob}$.

Similarly if $\infty' B$ meet OA in b', the value of $\{\infty' \,.\, AB\, \infty\infty'\}$ or

$$\{Ab' \infty_{,} O\} \text{ is } \frac{Ab'}{\infty_{,} b'} \div \frac{AO}{\infty_{,} O} \text{ or } \frac{Ab'}{AO}.$$

Hence $\frac{Ab}{Ob} = \frac{Ab'}{AO}$; and therefore $\frac{OA}{Ob} = \frac{Ob'}{OA}$.

Thus any central radius of a hyperbola is a mean proportional between the central distances of the points where it is met by parallels to the asymptotes drawn from any point on the hyperbola.

Orthogonal Projection.

220. Let P be a point in space and Pp the perpendicular from P to a given plane. Then the point p is the *orthogonal projection* of P on this plane.

We shall use a large and small letter to denote a point and its projection. Thus if P move along a straight line $ABC\ldots$, p moves along a straight line $abc\ldots$ similarly divided to $ABC\ldots$ That is, the projection of a straight line is a straight line similarly divided. So the projection of a curve QRS and chord QS are a curve qrs and chord qs, and when Q, S coincide, so do q, s. Thus the tangent at Q projects into the tangent at q. A system of parallel straight lines projects into a system of parallel lines in one plane, and the lengths of *finite* parallel lines are by projection altered in the same ratio. A pencil of lines PA, PB, PC, PD becomes a pencil pa, pb, pc, pd of the same anharmonic ratio (for any transversal $ABCD$ projects into a similarly divided transversal $abcd$). A plane curve projects into a plane curve of the same degree. For let X, Y be co-ordinates of P referred to OX, OY, and x, y those of p referred to ox, oy. Then $X = \lambda x$, $Y = \mu y$, λ and μ being constants, so that the curve $f(X, Y) = 0$ projects into the curve $f(\lambda x, \mu y) = 0$, which is of the same degree. A conic projects into a conic, an ellipse into an ellipse, a parabola into a parabola, a hyperbola into a hyperbola.

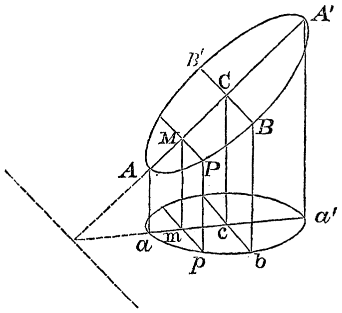
A diameter, being the locus of the middle points of a series of parallel chords, projects into a diameter, and therefore the centre of a conic (being the intersection of two diameters) into the centre of the projection. A circumscribing parallelogram becomes a circumscribing parallelogram.

In all this the lines of projection need not have been at right angles to the plane of projection, provided that they had been parallel. Such a projection had been *oblique*, but our projection will be orthogonal.

221. Let AA', BB' be the major and minor axis of an ellipse whose plane is inclined at an angle θ to the plane of

projection, the ellipse being so situated in its own plane that BB' is parallel to the plane of projection.

Then any ordinate PM is equal and parallel to its projection pm, and $\frac{am}{AM} = \frac{a'm}{A'M} = \frac{ca}{CA}$, and $cb = CB$.



Whence, as $\frac{PM^2}{AM \cdot A'M} = \frac{BC^2}{AC^2}$, $\frac{pm^2}{am \cdot a'm} = \frac{bc^2}{ac^2}$; so that the curve apa' is an ellipse with centre c and semi-axes ca, cb (Art. 122).

Also $ca = cb$, provided that $CA \cos\theta = CB$ (or $\sin\theta = e$).

Thus any ellipse can be projected orthogonally into a circle equal to its own lesser auxiliary circle.

And similarly a circle can be projected into an ellipse of any shape, and the circle will be equal to the greater auxiliary circle of the projection.

This method is employed to reduce theorems on the ellipse to theorems on the circle. For instance, Ex. 27 of Chapter IX. can be translated thus: If qv be perpendicular to a diameter pp' of a circle (whose centre is c), $qv^2 = pv \cdot vp'$, and if the tangent at q meet cp in t, then $cv \cdot ct = cp^2$.

For $\frac{QV^2}{CD^2} = \frac{qv^2}{cd^2}$ and $\frac{PV \cdot VP'}{CP^2} = \frac{pv \cdot vp'}{cp^2}$.

222. If any point P move round the contour of a plane area, p will move round the contour of an area which will be equal to

the former area multiplied by the cosine of the angle between the planes of the areas. This is proved by dividing the two areas by planes indefinitely near to each other and perpendicular to the common section of the planes of the areas. Areas lying on parallel planes are proportional to their projections, even when these are oblique.

Any projection made by parallel lines may be called *isometrical*, as preserving unaltered the ratios of lengths measured in a common direction and the ratios of areas lying in parallel planes.

223. The rules of this method of projection can also be established without geometry of three dimensions.

For let a point P be referred to co-ordinate axes, and let the projection of P be defined as a point P' in the plane of reference whose co-ordinates are obtained from those of P by altering the x of P a constant ratio, and the y in a constant ratio; or, λ and μ being constants, let the projection of the point (x, y) be defined as the point $\left(\frac{x}{\lambda}, \frac{y}{\mu}\right)$: then results may be obtained similar to those of Art. 221, the curve $\phi(xy)=0$ projecting into

$$\phi(\lambda x, \mu y)=0.$$

Thus, by making $\lambda=\frac{a}{c}$ and $\mu=\frac{b}{c}$, we project the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ into the circle $x^2+y^2=c^2$ (e.g. see fig. to Art. 123).

By making $\lambda=\frac{a}{c}$ and $\mu=\frac{b}{c}\sqrt{-1}$, we project the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ into the same circle, and thus we may extend some results from the circle to the hyperbola.

Ex. To find the equation to the tangent to the curve

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

at the point $x'y'$.

Projecting the curve into a circle, we have to find the tangent to $x^2+y^2=c^2$ at the point $\frac{x'}{\lambda}, \frac{y'}{\mu}$. This is

$$x \cdot \frac{x'}{\lambda} + y \cdot \frac{y'}{\mu} = c^2.$$

And this line is the projection of

$$\frac{x}{\lambda} \cdot \frac{x'}{\lambda} + \frac{y}{\mu} \cdot \frac{y'}{\mu} = c^2,$$

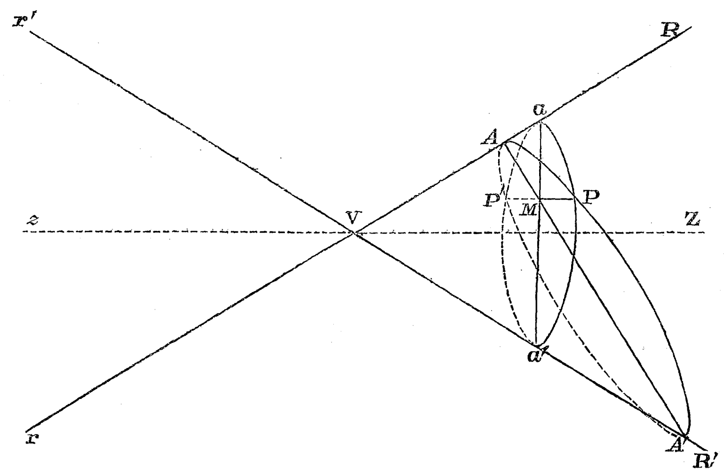
or of
$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

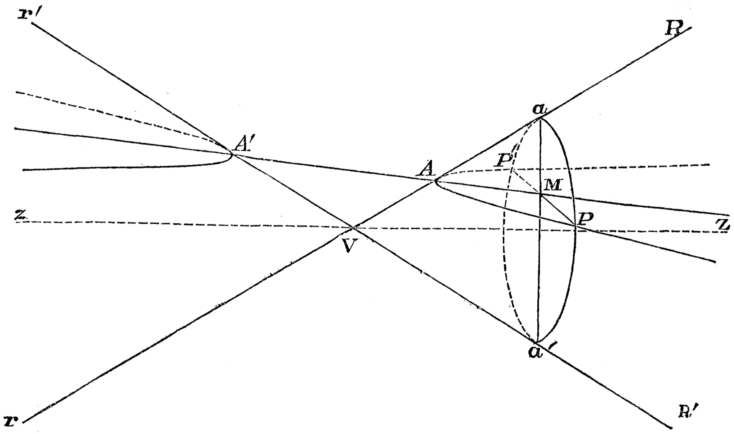
since
$$\lambda = \frac{a}{c} \text{ and } \mu = \frac{b}{c}\sqrt{-1}.$$

We can project a conic into any confocal.

Sections of a Cone.

224. Let a plane figure consisting of two straight lines ZVz, RVr inclined at an angle α revolve about one of the lines Zz. The surface generated is a '*right circular*' *cone* (or cone of revolution) with *vertex* Vz and *axis* Zz, and *semi-vertical angle* α. It





consists of two equal and similar branches, or semi-cones, which are infinite. Any plane meets this conical surface in a plane curve which, as we shall see, is a *conic section* in our sense of that term. For instance, all sections perpendicular to the axis are circles (Euclid, I. 26) centred in the axis.

A plane section through V perpendicular to the axis is a point-circle.

Let a section of the cone be made by *any* plane ϖ meeting the cone in a curve λ. Through Zz draw a plane at right angles to ϖ meeting the cone in the 'generating lines' Rr, $R'r'$, the plane ϖ in the line AA', and the curve λ in the points A, A'. Through any point P of λ draw a plane perpendicular to Zz, meeting VR, VR', AA' in a, a', M. This plane meets the cone in a circle of which aa' is a diameter, and the common chord PP' of the circle and λ is (Euclid, XI. 19) perpendicular to the plane RVR' and therefore to aa', AA'. Thus aa', and therefore AA', bisects P' at right angles, and therefore AA' divides λ symmetrically.

Now for all positions of P on λ the ratios

$$\frac{aM}{AM}, \quad \frac{a'M}{A'M}$$

are constant.

Also (Euclid, III. 35) $PM^2 = aM \cdot a'M$.

Therefore $\dfrac{PM^2}{AM \cdot A'M} = \dfrac{aM \cdot a'M}{AM \cdot A'M} =$ a constant.

Thus the locus of P is a 'conic section' of which AA' is the transverse axis.

If A, A' lie on the same semi-cone, λ is an ellipse; if A, A' lie on different semi-cones, λ is a hyperbola. If A' be at infinity, that is, if AA' be parallel to Rr,

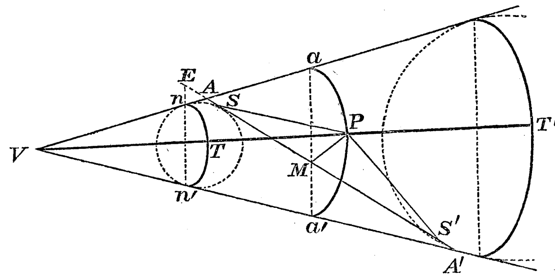
$$\frac{PM^2}{AM} = \frac{aM \cdot a'M}{AM}$$

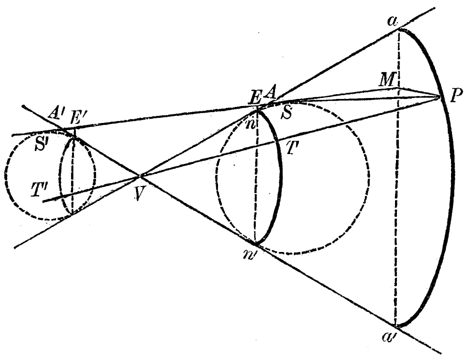
which is constant as $\dfrac{aM}{AM}$ and $a'M$ are constant. In this case the section is a parabola.

Let VH, $VH_{,}$ be any two generating lines of the cone, and let $VH_{,}$ move up to coincidence with VH. The points h, $h_{,}$ where VH, $VH_{,}$ meet *any* fixed plane section ultimately coincide and $hh_{,}$ is ultimately a tangent at h to that section and therefore to the cone. Thus the plane $VHH_{,}$ ultimately contains all the tangent lines that can be drawn to the cone at points of VH, and is the tangent plane to the cone 'along VH.' It is perpendicular to the plane VHZ, since it contains tangent lines to circular sections and these tangent lines are perpendicular to VH and VZ. Thus the plane of the parabolic section aforesaid is parallel to the tangent plane along VR' (Euclid, XI. 18), for a line in the plane RVR' perpendicular to the parabola's axis is perpendicular to both planes.

225. In general, if spheres inscribed in the cone so as to touch ϖ have S, S' for their points of contact with ϖ, then S, S' are foci of λ. For the spheres have circles of contact with the cone, and if VP meet these circles in T, T', TT' is constant. But $TT' = SP \pm S'P$ (according as the section is an ellipse or a hyperbola), because the two tangents PT, PS to one sphere are equal, as also the two tangents PT', PS' to the other sphere. Thus S, S' are foci of λ. Again, the planes of contact meet ϖ

in the *directrices*. For let the plane of contact of the sphere S meet VA, VA', AA' in n, n', E, and the plane ϖ in the line EK, and let a, a', be as before. Then EK being (Euclid, XI. 16) parallel to PM, ME is P's distance from EK. Now ME bears





(Euclid, VI. 2) a constant ratio to an or PT or SP. Thus the curve λ has S for focus and EK for directrix, the eccentricity being $\dfrac{an}{ME}$ or $\dfrac{An}{AE}$, which $=\dfrac{\sin AEn}{\cos\alpha}$.

Thus if e be the eccentricity of a section whose plane makes an angle θ with the planes of circular section,

$$e=\frac{\sin\theta}{\cos\alpha}.$$

If the section be a hyperbola the spheres are in different semi-cones. If the section be a parabola, AA' is parallel to a generating line Rr and S' is at infinity, but S is still the focus and E the foot of the directrix, as may be proved independently or inferred from *continuity*.

226. It follows from the geometry of the triangle VAA' that $AS.A'S$ $(= \sin^2\alpha . VA . VA')$ = the product of perpendiculars from A, A' on Zz. Thus *the semi-conjugate axis is a mean proportional between the distances of the extremities of the transverse axis from the axis of the cone.*

Again, the semilatus-rectum $\left(\text{or } \frac{b^2}{a}\right)$

$$= \frac{2\sin^2\alpha . VA . VA'}{AA'}$$

$$= \frac{\tan\alpha . VA . VA' \sin 2\alpha}{AA'}$$

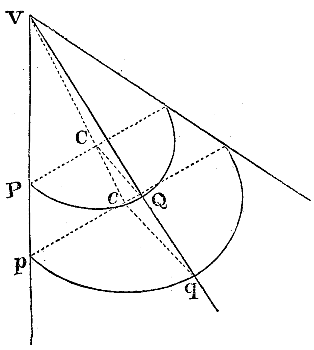
$$= \frac{2\tan\alpha . \Delta VAA'}{AA'}$$

$$= \tan\alpha \times \text{distance of } V \text{ from the plane of section},$$

and varies, therefore, as that distance.

227. All parallel sections of the cone are similar and, if we may apply the term to figures in different but parallel planes, similarly placed. Any line through V meets their planes in corresponding points.

For let two generating lines VPp, VQq and any line VCc through V meet two parallel planes of section in P, Q, C, p, q, c.



Then CP, CQ are parallel to cp, cq (Euclid, XI. 16), and therefore the angles QCP, qcp are equal (XI. 10). And by similar triangles

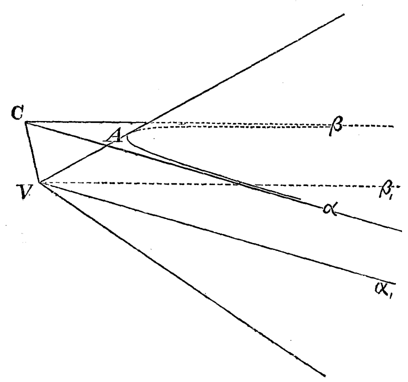
$$\frac{cp}{CP}=\frac{cV}{CV}=\frac{cq}{CQ}.$$

Thus the curves PQ, pq are similar and similarly placed, and C, c, regarded in connexion with them, are corresponding points. Thus if C be the centre of PQ, c is the centre of pq, and if CP, CQ be conjugate semi-diameters, so are cp, cq.

Let a section be drawn through *any* chord of the cone, and let PP' be the parallel diameter in the section, and QQ' the conjugate diameter. Then the plane VQQ' bisects all chords in the cone which are parallel to PP', and the plane VPP' bisects all chords parallel to QQ'.

Again, if $O\alpha$, $O\beta$ be asymptotes of any section, the asymptotes of any parallel section are parallel to $O\alpha$, $O\beta$ and lie in the planes $VO\alpha$, $VO\beta$.

Among such asymptotes, by moving the plane of section

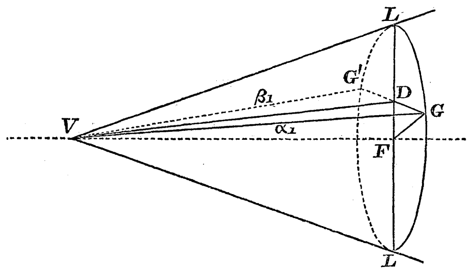


parallel to itself up to V, we are enabled to reckon the lines $V\alpha_1$, $V\beta_1$, which constitute the section through V.

228. From this section through V we may deduce the formula

$$e=\frac{\sin\theta}{\cos\alpha}\qquad\qquad (1).$$

For let a plane perpendicular to VZ meet VZ, $V\alpha_1$, $V\beta_1$ in



F, G, G', and let LL' a diameter of the circle bisect GG' in D. Then the eccentricity of the section $\alpha_1 V\beta_1$ is the same as that of any parallel section and

$$= \sec \frac{GVG'}{2}$$

$$= \frac{VG}{VD} = \frac{VF \sec FVG}{VF \sec FVD} = \frac{\sin\theta}{\cos\alpha}.$$

So long as we keep to the same circle LGL', VG is constant and $e \propto \dfrac{1}{VD}$.

Suppose D to coincide with F. Then $e = \sec\alpha$ and is the greatest possible. If D move from F towards L or L', e continually decreases, and when D reaches L or L' the lines VG, VG' coincide and the section is parabolic. When D has passed beyond the circle, the lines VG, VG' are no longer real (they are a point-ellipse), but the formula (1) still holds good. The section is now an ellipse whose eccentricity continually decreases through all proper fractions down to zero. The section is then a circle.

229. Any given conic which is not rectilinear and whose eccentricity does not exceed $\sec\alpha$ can be placed so as to coincide with some section of our cone. For let the given eccentricity be e. Draw a plane inclined to the planes of circular section at an angle $\sin^{-1}(e\cos\alpha)$. This will meet the cone in a section

similar to the given conic: and by properly choosing the distance of the plane from V we get a section of the right size.

A given conic, once fitted to the cone, would be capable of revolving round the cone's axis without breaking its coincidence.

We can of course cut from the cone two straight lines intersecting at any angle less than α.

If we wish for a section of greater eccentricity than $\sec\alpha$, we must take a cone of greater vertical angle. As the proposed eccentricity increases, this cone expands and tends to become two coincident planes.

230. The eccentricity of a rectilinear conic may be ambiguous. Thus two straight lines intersecting at an angle β are a hyperbola whose eccentricity is either $\sec\frac{\beta}{2}$ or $\sec\frac{\pi-\beta}{2}$. Two parallel straight lines are either a parabola or an infinitely eccentric hyperbola. For they are the limit either of a parabola having a given axis and passing through a given point, or of a hyperbola of given transverse axis and infinitely increased eccentricity. Two coincident lines are always parabolic.

In any given straight line take a finite portion AA'. Describe an ellipse and hyperbola having AA' for transverse axis, and let their eccentricities approach indefinitely near to unity. The ellipse approaches to coincidence with the finite line AA', the hyperbola to coincidence with the remaining infinite portion of the given line.

231. If a circular section of the cone be fixed and the vertex move off to infinity, the cone becomes a right circular cylinder.

The equation $e=\frac{\sin\theta}{\cos\alpha}$ is now $e=\sin\theta$. Thus e can have any value from 0 to 1.

A parabolic section consists of two parallel straight lines, the distance between which cannot exceed the diameter of the circular sections.

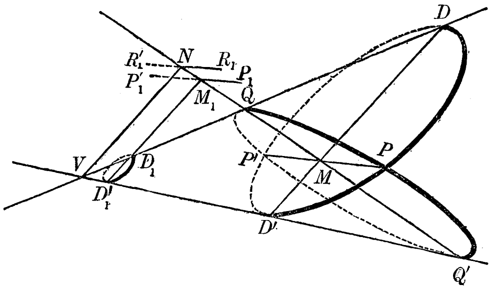
Oblique Cones.

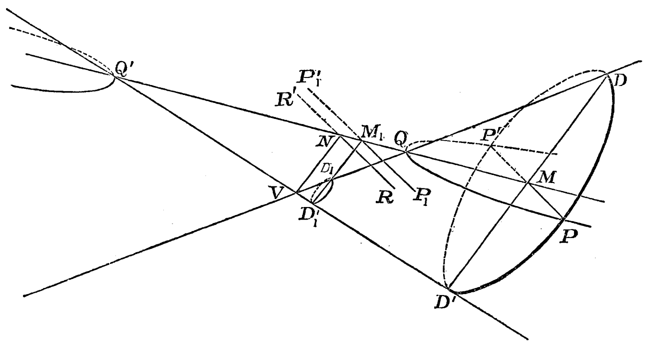
232. Let V be a given point, and Λ a given curve of the second degree lying in a plane Π. A straight line passing through V and Λ generates a cone which in general is not a cone of revolution, but an *oblique* cone. The curve Λ which guides the moving line may be called the *directrix.*

All parallel sections of such a cone are similar and similarly placed (cf. Art. 227). Thus all sections by planes parallel to Π are similar and similarly placed to Λ. By making sections through the vertex V we may conclude *à priori* that sections meeting both semicones are hyperbolas and that sections parallel to a tangent plane are parabolas. But we shall directly prove that the section λ made by any plane ϖ is a conic.

At any point P of λ draw a section Λ_0 by a plane Π_0 parallel to Π, and let PP' be the common chord of λ, Λ_0.

As P moves along λ, PP' moves parallel to itself (Euclid, XI.





16). In Π_0 draw a diameter DD' bisecting PP' in M. The plane VDD' bisects all chords in the cone that are parallel to

PP' and is fixed; and, as P moves, the line DD' moves parallel to itself in this fixed plane. (See Art. 227.) Therefore, if VD, VD' meet λ in Q, Q', all chords of λ that are parallel to PP' are bisected by the line QQ', and the ratios $\frac{DM}{MQ}$, $\frac{D'M}{MQ'}$, are constant. Again, if $k \,.\, DD'$ be the length of that diameter in Λ which is conjugate to DD', k is constant, and $\frac{PM^2}{DM \,.\, D'M} = k^2$. Therefore $\frac{PM^2}{QM \,.\, Q'M} =$ a constant (k'^2), and so λ is a conic in which QQ' is a diameter conjugate to PP'.

The cone may be described with V as vertex and the curve λ as directrix. Indeed any curve drawn upon the cone so as to cross all the generating lines is a possible directrix.

233. Let any section Λ_1 be made by a plane Π_1 parallel to Π_0 which meets QQ' in a point M_1 lying without λ. Then if P_1P_1' be the line in which Π_1 meets ϖ, P_1P_1' is parallel to PP' and therefore conjugate to QQ': and if VD, VD' meet Π_1 in D_1, D_1', the diameter D_1D_1' of Λ_1 is conjugate to P_1P_1', and $k \,.\, D_1D_1'$ is the length of Λ_1's diameter parallel to P_1P_1'. Also by similar triangles

$$\frac{QM_1 \,.\, Q'M_1}{D_1M_1 \,.\, D_1'M_1} = \frac{QM \,.\, Q'M}{DM \,.\, D'M} \text{ and therefore } = \frac{k^2}{k'^2}.$$

Let a line through V parallel to DD' meet QQ' in N, and let RNR' be parallel to PP'. Then if Π_1 move up to V, the points D_1, D_1' coincide, the section Λ_1 becoming a point-section in the plane VRR'; and M_1 is at N. And thus

$$\frac{QN \,.\, Q'N}{VN^2} = \frac{k^2}{k'^2}.$$

234. In the special case in which Λ is a circle, $k = 1$ and DD' is perpendicular to PP'. Thus VN is perpendicular to RR' and $= k' \sqrt{QN \,.\, Q'N}$. The section made by the plane VRR' is a point-circle and the plane Π is parallel to this plane.

Hence, given any conic λ and a line RR' lying in its plane but not intersecting it, we can determine a point V, such that the

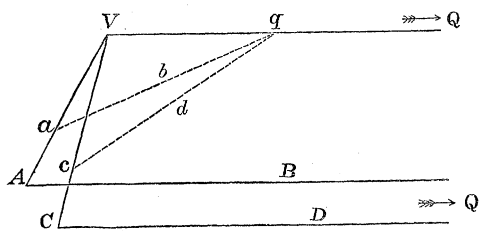
cone with vertex V and directrix λ shall be met in a circle by any plane parallel to the plane VRR'.

For draw a diameter QQ' of λ conjugate to RR' and meeting RR' in N, and with centre M and radius $\sqrt{k'.QN.Q'N}$ ($k'.QQ'$ being the length of the diameter conjugate to QQ') describe a circle in a plane perpendicular to RR'. The point V may be any point on this circle.

Perspective Projection.

235. Let V be a given point or *vertex* in space, and let ϖ be a given *plane of projection*. Let the line joining V to any point P meet ϖ in p. Then p is the perspective projection of P on the plane ϖ. Any system of points $P, Q, R \dots$ has a perspective projection (or picture) $p, q, r \dots$ on this plane. If P move along a straight line AB, p moves along a straight line ab, which is the intersection of the planes ϖ, VAB.

Let AB, $CD, \dots$ be parallel straight lines which are to be projected.



Through V draw a parallel Vq meeting ϖ in q. Then the planes VAB, $VCD, \dots$ all pass through Vq, and therefore the lines ab, $cd, \dots$ all pass through q. The point q (which is the projection of Q the point at infinity on the given parallels) is at an infinite distance (or, ab, $cd, \dots$ are parallel) only when the lines AB, $CD, \dots$ are parallel to ϖ.

A system of lines meeting in a point L project into parallels if ϖ be parallel to VL. If another system meet in M, and ϖ be taken parallel to the plane VLM, the two systems project into two

sets of parallels. Thus, if L, M be two vertices of any quadrilateral, the quadrilateral projects into a parallelogram on any plane parallel to VLM. The line LM is thus projected to infinity.

Curves, chords, tangents and points of contact project into curves, chords, tangents and points of contact.

By Art. 232 the projection of a conic is another conic.

The anharmonic ratio of a pencil PA, PB, PC, PD is unaltered by projection, for

$$\{P.\,ABCD\} = \{ABCD\} = \{V.\,ABCD\} = \{V.\,abcd\}$$
$$= \{abcd\} = \{p\,.\,abcd\}.$$

Thus if P be the pole of QR with reference to a conic, p is the pole of qr with reference to the conic's projection. If QR be projected to infinity, P is projected into the centre.

By Art. 234, given any conic λ and any line RR' in its plane lying without it, we can choose V so as to project λ into a circle and the line RR' to infinity: or so as to project λ into a circle and a given point (the pole of RR') in λ's plane and within λ into the centre of the circle.

236. By simply projecting the conic into a circle we may extend many theorems on the circle to the other conic sections, as for instance those of Arts. 88, 89, 92, 101, and Pascal's Theorem. Such properties are called *projective*.

Again, we may sometimes make with advantage a more liberal use of the method, as for instance to prove the theorem: *If two triangles be self-conjugate with regard to an ellipse, their angular points lie in a conic.*

Let ABC, DEF be the two triangles. Project to infinity that side EF of the triangle DEF which lies without the ellipse, and project the ellipse at the same time into a circle. Then d is the centre of the circle, and de, df are at right angles. Also abc is a triangle self-conjugate with regard to the circle. Thus da, db, dc are perpendicular to bc, ca, ab, since the line joining the centre of a circle to the pole of a given straight line is perpendicular to the straight line. (Ferrers' *Trilinear Co-ordinates.*)

Now a conic passing through a, b, c, and the two points e, f at infinity will be a rectangular hyperbola, since its asymptotes are parallel to de, df, and will therefore pass through the orthocentre of the triangle abc, that is, through d. Thus a, b, c, d, e, f lie on a conic, and therefore A, B, C, D, E, F lie on a conic.

237. When V moves off to infinity, the lines of projection become parallel and the projection isometrical. There is also an analytical view of the connection of the isometrical and perspective methods*.

Let AB be a fixed line of reference in a plane ϖ, and P, P' any points in the plane, and let PM, $P'M'$ be the perpendiculars from P, P' to AB. Then in the isometrical projection on a plane Π, pm, $p'm'$ are not necessarily perpendicular to ab, but if pn, $p'n'$ be perpendicular to ab,

$$\frac{pn}{p'n'} = \frac{pm}{p'm'} = \frac{PM}{P'M'},$$

and therefore $\dfrac{PM}{pn}$ is the same for all points in ϖ.

Let X, Y be co-ordinates of P referred to AB as the axis of x, and to any other line in ϖ as the axis of y; x, y co-ordinates of p referred to any axes whatever in Π; and let X', Y', x', y' be corresponding attributes of P': then if the line ab be denoted by

$$ax + by + c = 0,$$

$$\frac{PM}{PM'} = \frac{Y}{Y'}, \text{ and } \frac{pn}{pn'} = \frac{ax + by + c}{ax' + by' + c},$$

so that
$$\frac{Y}{ax + by + c} = \frac{Y'}{ax' + by' + c};$$

and thus no motion of P in ϖ affects $\dfrac{Y}{ax + by + c}$.

* Much of this Article is due to Mr W. K. Clifford of Trinity College, Cambridge.

Thus we may assume

$$Y = A_2x + B_2y + C_2 \text{ .... } (1),$$

A_2, B_2, C_2 being the same for all points in ϖ.

Similarly we may assume

$$X = A_1x + B_1y + C_1 \text{ .... } (2),$$

A_1, B_1, C_1 being the same for all points in ϖ.

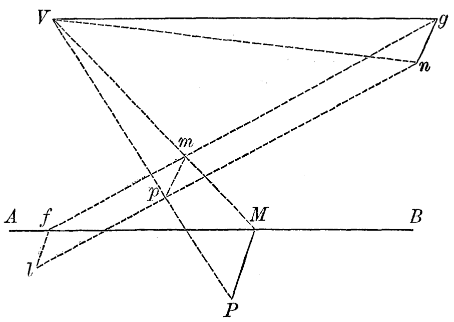
Thus the curve $\phi\,(XY) = 0$ projects into

$$\phi\,(A_1x + B_1y + C_1,\ A_2x + B_2y + C_2) = 0,$$

and the point XY into a point xy determined by equations (1), (2).

Again, let the projection be perspective, and let fl, gn be the intersections of Π with ϖ and with a plane through V parallel to ϖ; so that fl, gn are parallel. Let AMB be a line of reference in ϖ, and let the plane VAB meet the planes ϖ, Vgn in fg, Vg. Then Vg is parallel to AB.

Let P be any point in ϖ, and PM a parallel to lf, and let



p, pm be projections of P, PM; so that pm, PM, fl, gn are parallel.

Draw lpn parallel to fg. Then

$$\frac{PM}{pm} = \frac{VM}{Vm} = \frac{gf}{gm} = \frac{gf}{pn}.$$

If P' be any other point in ϖ, and letters be accented accordingly,

$$\frac{PM'}{p'm'} = \frac{gf}{p'n'},$$

for g, f do not change. Therefore

$$\frac{PM}{pm \div pn} = \frac{P'M'}{p'm' \div p'n'},$$

and is therefore not affected by any motion of P in ϖ.

Let X, Y, X', Y' be co-ordinates of P, P' referred to AB as axis of X, and to any other line in ϖ as axis of Y, and x, y, x', y' co-ordinates of P, P' referred to any axes in Π, and let fg, fl referred to these last axes be

$$lx + my + n = 0, \quad fx + gy + h = 0.$$

Then

$$\frac{PM}{P'M'} = \frac{Y}{Y'},$$

$$\frac{pm}{p'm'} = \frac{lx + my + n}{l'x + m'y + n},$$

and

$$\frac{pn}{p'n'} = \frac{fx + gy + h}{fx' + gy' + h}.$$

Therefore

$$\frac{Y}{\dfrac{lx + my + n}{fx + gy + h}},$$

being equal to

$$\frac{Y'}{\dfrac{lx' + my' + n}{fx' + gy' + h}},$$

is the same for all points in ϖ, and therefore

$$Y = \frac{L_2x + M_2y + N_2}{Lx + My + N} \quad \text{(1)},$$

L_2, M_2, &c. being the same for all points in ϖ.

Similarly

$$X = \frac{L_1'x + M_1'y + N_1'}{L'x + M'y + N'},$$

L_1', L', &c. being the same for all points in ϖ.

Now L', M', N' are, like L, M, N, in the proportion of f, g, h, for however AB be changed, gn does not change.

Therefore we may assume

$$L' = kL, \quad M' = kM, \quad N' = kN.$$

If then L_1, L_2, L_3 denote

$$\frac{L_1'}{k}, \ \frac{L_2'}{k}, \ \frac{L_3'}{k},$$

$$X = \frac{L_1x + M_1y + N_1}{Lx + My + N} \text{..................... (2).}$$

And if U, V, W denote

$$L_1x + M_1y + N_1, \quad L_2x + M_2y + N_2z, \quad Lx + My + Nz,$$

$\phi(X, Y) = 0$ projects into

$$\phi\left(\frac{U}{W}, \ \frac{V}{W}\right) = 0,$$

and the point XY into a point xy determined by the equations (1), (2).

The line

$$pX + qY + r = 0 \text{........................(3)}$$

projects into

$$pU + qV + rW = 0,$$

which is at infinity if

$$\left.\begin{array}{l} pL_1 + qL_2 + rL = 0 \\ pM_1 + qM_2 + rM = 0 \end{array}\right\} \text{....................(4).}$$

The conic

$$aX^2 + bY^2 + c + 2a'Y + 2b'X + 2c'XY = 0 \text{.........(5)}$$

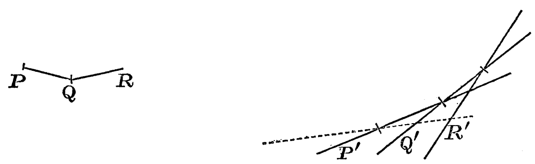
projects into

$$aU^2 + bV^2 + cW^2 + 2a'VW + 2b'WU + 2c'UV = 0 \ldots\ldots(6).$$

If L_1, M_1, ... be so chosen that (6) may be a circle and the conditions (4) be satisfied, then the given conic is projected into a circle and at the same time a given line to infinity. Theoretically the analytical operation is always possible, but our previous geometrical experience (Art. 234) assures us, that if the line cut the conic in real points the values of L, M, ... cannot be all real. When the projection is isometrical, L, M are zero.

The Method of Reciprocal Polars.

238. Let P be any point, and P' its polar with regard



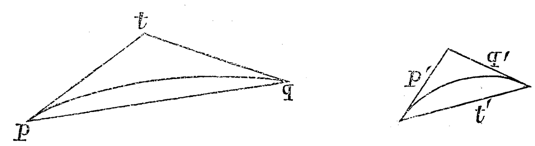
to a given conic section. Then if P move to some new position Q, P' will move to some new position Q'. Also the point in which P', Q' intersect, or 'the point $(P',\ Q')$,' will be the pole of the line PQ (cf. Art. 101). From Q let the moving point pass to R and the moving line to R'. Then the point $(Q',\ R')$ is the pole of the line QR. And so on.

Ultimately, when the stages of the point's motion are indefinitely small, we have two curves, one described by the point and the other by the intersection of consecutive lines. To the first curve PQ is ultimately a tangent at P, and to the second curve P' is ultimately a tangent at the ultimate position of (P', Q') (Art. 83: for the points in which P' is met by its predecessor and by Q' are ultimately on the second curve).

Since then, in the *rectilinear* figures, P and PQ have their correspondents P' and (P', Q'), we see that in the *curves* a point and its tangent have a corresponding tangent and point of contact.

From the reciprocity of their relation the curves are called *reciprocal* polars with respect to the given conic.

Call the curves A, A', and let tp, tq be two tangents to A at p, q. Then corresponding to p, q, the chord pq, the tangents pt,



qt, and their intersection t, the curve A has the tangents p', q', their point of intersection (p', q'), the points of contact (p', t'), (q', t'), and the chord of contact t'.

Thus two tangents and a chord of contact 'reciprocate into' two tangents and a chord of contact, though not respectively.

239. If a point P move along a straight line Q, its polar P' passes through a fixed point Q'*. If P describe a curve, P' envelopes a curve.

Corresponding to a triangle XYZ we have a triangle with sides X', Y', Z', and by taking the right positions for X, Y, Z we can thus produce *any* copolar triangle.

If the curve A circumscribe a triangle, the curve A' touches the three sides of the copolar triangle, and if A touch the three sides of a triangle, A' circumscribes the copolar triangle.

The reciprocal of a conic section with respect to a conic section is also a conic section. For draw analytically any tangent to the first curve and take its pole with regard to the second: the locus of this pole will be found to be of the second degree.

To a conjugate triad FGH corresponds a conjugate triad. For if GH is the chord of contact to F, the point (G', H') has F' for its chord of contact, &c.

240. Now let us take the theorem: *If there be two triangles* ABC, abc, *such that* Aa, Bb, Cc *meet in a point, then the points*

* On the working notation of this subject compare a suggestive note in Ferrers' *Trilinear Co-ordinates*, Chap. VI.

(BC, bc), (CA, ca), (AB, ab) *lie in a straight line* (1). By the method of Reciprocal Polars we can deduce another theorem from this.

For, reciprocating with respect to any conic, we have, for the triangles ABC, abc such that ..., *two triangles with sides* A′, B′, C′, a′, b′, c′, *such that* (A′, a′), (B′, b′), (C′, c′) *lie in a straight line.* Also for the points (BC, bc), &c., we have *the lines joining* (B′, C′), (b′, c′), *&c.* These three lines therefore meet in a point.

Therefore, *If there be two triangles such that the intersections of corresponding sides be in a straight line, the lines joining corresponding vertices meet in a point.* (2)

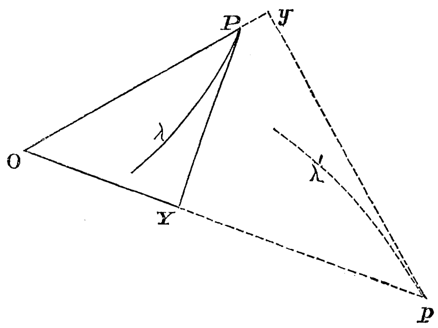
The *generality* of (2) is fairly established: for any case of (2)'s hypothesis has by reciprocation a corresponding case of (1)'s hypothesis, and (1)'s conclusion, following that case, draws after it the conclusion of (2).

241. When the conic of reference is a circle, one of the angles between any two lines is equal to the angle subtended at the centre of the circle by their poles. Thus if two lines be at right angles their poles subtend a right angle at the centre, and conversely.

Take now the theorem: 'The circles described on the three diagonals of any quadrilateral have a common chord.' This, first expressed in the form 'At a point where two diagonals AC, BD of a quadrilateral $ABCD$ subtend a right angle the third diagonal also subtends a right angle,' can be translated into a much simpler but equivalent theorem. For reciprocate with respect to a circle centred at the point, or (as we say) 'reciprocate with respect to the point.' The result is found to be:

If A', B', C', D' be four lines such that A' is at right angles to C', and B' at right angles to D'; then the line joining the points (A', B'), (C', D') is at right angles to the line joining the points (A', D'), (B', C'); which is the first theorem of Art. 53.

242. Our curve of reference will henceforth be a circle and the centre of this circle of reference will be called simply 'the centre,' and if k be its radius, we shall say that k^2 is the 'constant of reciprocation,' suppressing all mention of the circle of reference. The reciprocal λ' of a curve λ with respect



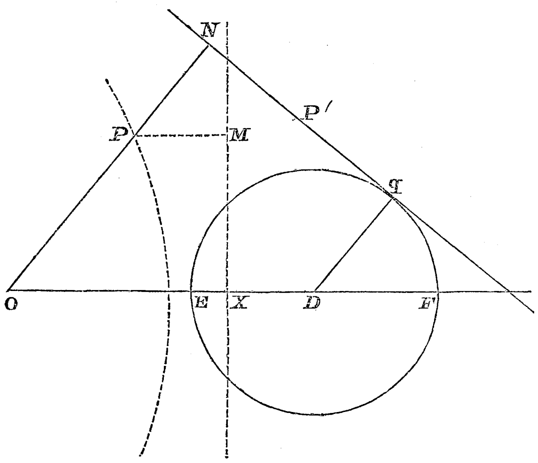
to a point O can be found in two ways. For take any point P in λ, and in OP take y, so that $OP \cdot Oy = k^2$. Then λ' is the *envelope* of a line through y perpendicular to Op. Again, draw OY perpendicular to the tangent to λ at P, and in OY take p so that $Op \cdot OY = k^2$. Then λ' is the *locus* of p.

Also as the angle Oyp is a right angle (Euclid VI. 6), p is the point where the aforesaid line through y touches its envelope λ'. If λ be a conic, so is λ'.

If OL, OM be tangents from O to λ, then the line O' is at infinity, and L', M' are asymptotes to λ'.

Hence if O be *within* λ, since OL, OM cannot be drawn, L', M' are imaginary, and λ' is an ellipse. If O be without λ, the asymptotes of λ' are real and λ' is a hyperbola. If λ pass through O, λ' touches the line at infinity and is a parabola.

243. Let P be the pole of the tangent P' to a given circle at any point q. Then OP meets P' at right angles, say in N, and $OP \cdot ON = k^2$. Draw PM at right angles to D's polar (which



is a line MX perpendicular to OD and such that $OD.OX = k^2$).
Then (Art. 101)

$$\frac{P(D)}{D(P)} = \frac{O(D)}{O(P)},$$

or

$$\frac{PM}{Dq} = \frac{OX}{ON} = \frac{OP}{OD};$$

whence

$$\frac{PM}{PO} = \frac{\rho}{c}.$$

Thus λ' is a conic section having O for focus and the polar of D for directrix; and the eccentricity of λ' is $\frac{c}{\rho}$. The semi-latus rectum is $\frac{c}{\rho}.OX$, or $\frac{k^2}{\rho}$. Thus the formulæ for the 'elements' of the conic are

$$l = \frac{k^2}{\rho} \quad (1), \qquad e = \frac{c}{\rho} \quad (2).$$

From the tangents at E, F we derive the vertices of the conic. Thus the centre of the conic is at a distance

$$\frac{1}{2}\left(\frac{k^2}{c+\rho} + \frac{k^2}{c-\rho}\right) \text{ or } \frac{k^2 c}{c^2-\rho^2} \text{ from } O.$$

The asymptotes are lines through this point perpendicular to the tangents from O to S.

The reciprocal of a conic with respect to its focus is a circle whose centre is the pole of the directrix and whose radius is inversely proportional to the latus rectum.

244. As another example of the method of reciprocal polars we shall solve the problem:

'A triangle is inscribed in a given conic so that the focus is the centre of the triangle's inscribed circle. Find the radius of this circle.' (Senate-House Problems, 1862.)

Reciprocating the supposed construction with respect to the focus O, we have for the conic a circle of radius $\frac{k^2e}{l}$ whose centre O' is such that $OO' = \frac{k^2e}{l}$ (Equations 1, 2 of Art. 243); for the triangle a triangle described *about* this circle, and for the circle a circle described about the new triangle with centre O and radius $\frac{k^2}{x}$, if x be the quantity sought.

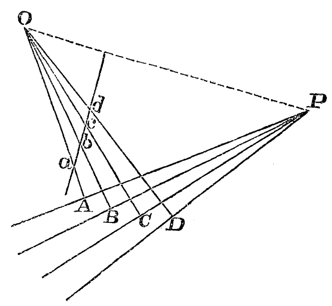
Thus O, O' are centres of the inscribed and circumscribed circles of the new triangle, and therefore by Trigonometry

$$OO'^2 = \left(\frac{k^2}{x}\right)^2 - 2\frac{k^2}{x}\cdot\frac{k^2}{l}.$$

Thus $$\left(\frac{e}{l}\right)^2 = \left(\frac{1}{x}\right)^2 - \frac{2}{xl} \quad \text{or} \quad x = \frac{l}{1+\sqrt{1+e^2}}.$$

245. If four straight lines meet in a point, their poles lie in a straight line. Also the anharmonic ratio of the lines is equal to that of the points.

For let a, b, c, d be poles of the four lines PA, PB, PC, PD.



Then the angles OAP, OBP, OCP, ODP are right angles, and the angles AOB, BOC, COD are (Euclid III. 21) equal to the angles APB, BPC, CPD. Thus the anharmonic ratio of the four lines through P is equal that of the four lines through O, which is that of a, b, c, d (Art. 65).

By projection we extend this theorem to cases in which the curve of reference is any conic.

And by reciprocating the theorem of Art. 218, we see that any tangent to a conic is cut by four fixed tangents in points whose anharmonic ratio is constant.

Examples on Chapter XIII.

1. Given five points on a conic, find by a geometrical construction the tangent at any one of them.

2. If A, B, C, D be four fixed points and P a point such that $\{P\,.\,ABCD\}$ is constant, the locus of P is a conic through A, B, C, D.

3. Hence prove that if two triangles circumscribe a conic, their angular points lie in another conic.

4. Apply the method of orthogonal projection to Exs. 17, 18, 20, 31, 32, 41, 51 of Chap. IX.

5. The equation to a curve referred to any triangle in its plane is $\phi(\alpha\beta\gamma) = 0$, and the equation to the projection of the curve referred to a triangle in the plane of projection is $\psi(\alpha\beta\gamma) = 0$. Prove that if $\alpha'\beta'\gamma'$ be the projection of any point $\alpha\beta\gamma$ in the first plane,

$$\frac{\phi(\alpha\beta\gamma)}{\psi(\alpha'\beta'\gamma')}$$

is constant for all points and for all triangles of reference in the first plane.

6. The sectorial areas included by the lines $y = 0$, $y = mx$ and the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are each $\dfrac{ab}{2}\tan^{-1}\dfrac{am}{b}$.

7. The area common to the circle $x^2 + y^2 = 2$ and the ellipse $x^2 + 3y^2 = 3$ is $\frac{\pi}{6}(4 + 3\sqrt{2})$.

8. No parallelogram circumscribing a given ellipse has a smaller area than one of those formed by tangents at the extremities of conjugate diameters.

9. In the ellipse the area of a sector whose bounding radii are conjugate semi-diameters is constant.

10. A chord is drawn to a given ellipse so as to cut off a constant area. Prove that this chord touches at its middle point a similar and similarly placed concentric ellipse.

11. TP, TQ are tangents to an ellipse whose centre is C. Prove that the points T, P, C, Q lie in a similar and similarly placed ellipse.

12. A plane section of a right cone being given, the locus of the vertex is a conic whose foci are the extremities of the transverse axis of the section (Arts. 122, 145, 225).

13. Prove that a section of the cone in Art. 232 parallel to a tangent plane is a parabola.

Also investigate independently that case of Art. 232 in which Λ is a parabola.

14. From a point V a perpendicular VN is drawn to a given plane, in which lies a circle of centre C, $NACB$ being the diameter through N.

Prove that the cone whose vertex is V and directrix this circle can have no section of greater eccentricity than $\sec \frac{AVB}{2}$, and that if VN' be drawn in the plane VAB making the same angles with VA, VB that VN makes with VB, VA, sections by planes at right angles to VN' are circles.

15. Prove by projection the Theorem of Chap. v. Ex. 22.

16. Given four points on a conic, the locus of the pole of a given line is a conic passing through the four points (Chap. XI. Ex. 37).

17. Apply perspective projection to Pascal's Theorem. Are all possible cases of the theorem accounted for in this proof?

18. Assuming the proposition of Chap. V. Ex. 23, deduce the converse.

19. Deduce a property of the circle from Euclid III. 31.

20. Prove from the properties of the circle that if parallel tangents be drawn to a conic the rectangle under their distances from the focus is constant.

21. Into what does the normal to a curve at a given point reciprocate? and what answers to a pair of conjugate diameters of a conic when the conic is reciprocated into a circle?

22. Translate Chap. VIII. Ex. 30 and Chap. X. Ex. 40 into propositions on the circle, and Chap. X. Ex. 30 into a proposition on the parabola.

Deduce from the circle the parabolic property in Chap. XII. Ex. 12.

23. If six tangents 1, 2, 3, 4, 5, 6 be drawn to a conic, the lines joining the points 12, 23, 34 respectively to the points 45, 56, 61 meet in a point (BRIANCHON'S THEOREM).

24. Confocal parabolas which have their axes coincident intersect at right angles if they intersect at all, and the focal distance of a point of intersection is an arithmetic mean between the semilatera recta.

CHAPTER XIV.

CURVES OF HIGHER DEGREES.

246. It has been implied in Arts. 21, 54, 100, 103, 177, that Analytical Geometry can deal with curves whose equations are of higher degrees than the second. Such equations have also appeared in the examples, as results of experiment in cases where loci had to be found, as for instance in Exs. 12, 13 of Chap. VIII.

In Ex. 7 of Chap. VII. four equations were given as subjects for the application of Art. 100. Of these (1), (2), (4) are of the 2nd, 3rd, and 6th degrees. The equation $\frac{y}{a} = \sin\frac{x}{b}$, like all equations that cannot be reduced to an algebraic form that is finite, rational, and integral, is called *transcendental.*

247. It is convenient to notice here certain properties of algebraic equations. The symbol $\equiv$ is a modification of $=$ and denotes *identical* equality, so that if we use $f(x)$ as an abbreviation for some expression involving x, we say that $f(x) \equiv$ that expression.

Let $\quad f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0 \ldots\ldots\ldots (1)$

be an equation of the n^{th} degree involving one variable x. This equation has n roots real or imaginary.

If imaginary roots occur, they occur in pairs: e.g. if $3 + 5\sqrt{-1}$ be a root occurring 7 times, $3 - 5\sqrt{-1}$ is also a root and occurs 7 times. If n be odd, there is one real root at least. Also if a root α occur r times, $f(x)$ is divisible by $(x-\alpha)^r$. And generally, if $\alpha_1, \alpha_2 \ldots \alpha_n$ be the roots,

$$f(x) \equiv p_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

Thus $\alpha_1 + \alpha_2 + \dots + \alpha_n = -\frac{p_1}{p_0}$, and $\alpha_1 \alpha_2 \dots \alpha_n = (-1)^n \frac{p_n}{p_0}$.

If $p_n = 0$, one of the roots is zero. If $p_{n-1} = 0$, two of the roots are zero. If the lowest power of x in the equation be the r^{th}, r of the roots are zero.

Again, $\frac{1}{x}$ is found from the equation

$$p_n \left(\frac{1}{x}\right)^n + p_{n-1} \left(\frac{1}{x}\right)^{n-1} + \dots + p_1 \left(\frac{1}{x}\right) + p_0 = 0,$$

in which, if p_0, p_1, ... p_{r-1} vanish, r roots are zero. Thus, if the coefficients of the r highest powers of x in (1) be indefinitely diminished, r of the roots become infinite. Any quadratic equation may be looked on as a cubic equation with an infinite root.

248. When we use a functional notation, such as $f(x)$, $\phi(x, y)$ and the like, it is to be understood, unless the contrary be stated, that the function is algebraic, rational and integral. Also a *factor* of a function is understood to involve a variable. Thus we do not say that $2x^2 + 4y^2 - 6$ breaks up into two factors, although 2 divides each coefficient.

Let $\phi(x, y) = 0$ be an equation of the n^{th} degree in two variables x, y. It is a *proper* equation of the n^{th} degree or an *improper* equation of the n^{th} degree according as $\phi(x, y)$ does not or does break up into factors. Thus

$$xy = x + y - 2$$

is a proper equation of the second degree, and

$$xy = x + y - 1$$

is an improper equation of the second degree.

249. To find all the pairs of values of x and y which satisfy the simultaneous independent equations

$$\phi(x, y) = 0 \quad \dots\dots\dots\dots\dots\dots (1),$$
$$Ax + By + C = 0 \quad \dots\dots\dots\dots\dots\dots (2),$$

we may eliminate y and determine the values of x from the resulting equation

$$\phi\left(x, -\frac{Ax+C}{B}\right) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If (1) be of the n^{th} degree, so is (3). Thus there are n values of x, real or imaginary, and for each value of x the linear equation (2) gives the value of y. The number of solutions of the system (1), (2) is therefore n.

If $Ax + By + C$ had been a factor of $\phi(x, y) = 0$, then (1), (2) would not have been independent, and (3) would have been an identical equation.

250. If $\psi(x, y) = 0 \ldots\ldots(1)$, $\phi(x, y) = 0 \ldots\ldots(2)$ be simultaneous independent equations of the m^{th} and n^{th} degrees, the number of solutions, real or imaginary, of the system is mn. The equations are *certainly* independent if each be proper; they *may* be independent though each be improper: but if $\psi(x, y)$, $\phi(x, y)$ have a common factor, the number of solutions is infinite; for any pair of values of x, y which make that factor vanish are a solution.

251. Returning to Art. 249, we see that the straight line represented by (2), if it be no part of the curve represented by (1), intersects that curve in n points, real or imaginary. If two values of x in (3) be the same quantity α, two of the n points coincide at the point $\left(\alpha, -\frac{A\alpha+C}{B}\right)$. Thus either the line touches the curve at this point, or the point is a double point on the curve (see Art. 83).

If r values of x in (3) be α, the point $\left(\alpha, -\frac{A\alpha+C}{B}\right)$ *may* be a multiple point of the r^{th} order.

If P be a multiple point of the 5th order on a curve of the 11th degree, a line drawn through P will in general meet the curve in 6 other points, real or imaginary. Suppose the line moved about P to a position in which 4 of these 6 points have moved up to P. Then the line has contact of the 4th order with the curve at P, and meets the curve in two other points.

If r values of x in (3) be α, the point P may be a multiple point of the r^{th}, $(r-1)^{\text{th}}$ ... 3rd, or 2nd orders, or may be no multiple point at all; and in these several cases the line is no tangent, or has contact of the 1st, 2nd ... $(r-2)^{\text{th}}$ or $(r-1)^{\text{th}}$ orders with the curve, respectively.

252. Again, the *curves* represented by the independent equations (1), (2) of Art. 250 have mn common points, real or imaginary.

Suppose one curve to pass through a multiple point (α, β) of the p^{th} order on the other, then p of the solutions of (1), (2) are $x=\alpha$, $y=\beta$. If the point (α, β) be a multiple point of the p^{th} order on one curve and the q^{th} order on the other, then since each branch of the second curve meets the first curve in p coincident points, *the point* (α, β) *counts as* pq *intersections*, and pq solutions of (1), (2) are $x=\alpha$, $y=\beta$. If the curves have contact of the r^{th} order at (α, β), then $r+1$ solutions of (1), (2) are $x=\alpha$, $y=\beta$. For instance, if P, Q, R be three points on a curve, and if, P remaining fixed, Q, R move up to coincidence with P, the circle PQR becomes ultimately the circle of curvature at P. Since we may suppose Q's motion to be completed before R's motion begins, a circle passing through R and always touching the curve at P will be the circle of curvature at P when R has moved up to P.

Or, since two touching curves have a common tangent at the point of contact, we may substitute for the words 'touching the curve at P,' touching at P the tangent to the curve at P.

253. If the curve $F(xy)=0$ pass through the origin, then since $F(0, 0)=0$, there is no constant term in $F(xy)$. Suppose $F(xy) \equiv ax+by+$ terms of 2, 3 ... n dimensions {which we may denote by $\phi_2(xy)$, $\phi_3(xy)$ ... $\phi_n(xy)$}. Then the polar equation is

$$0=(al+bm)\,r+\phi_1(lm)\,.\,r^2+\phi_2(lm)\,r^3+\ldots$$
$$+\phi_n(lm)\,r^n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This equation determines the n values of the radius vector

drawn in any given direction $[l, m]$. Two values of r are zero if

$$al + bm = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

That is, (2) is the polar equation to the tangent at the origin. The Cartesian equation is

$$ax + by = 0.$$

If a and b be zero, then for *any* given direction two of the roots of (1) are zero, that is, the origin is a *double* point. If however we move the radius vector into a position in which

$$\phi_2(l, m) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

then a third value of r becomes zero, and the vector-line is a tangent. That is, (3) is the polar equation to the tangent-lines at the origin. The Cartesian equation is

$$\phi_2(xy) = 0.$$

And generally, if

$$0 = \phi_r(xy) + \text{terms of higher dimensions} + \ldots + \phi_n(xy)$$

be the equation to the curve, the origin is a *multiple* point of the r^{th} order, and the r tangent lines at the origin are given by the equation

$$\phi_r(xy) = 0.$$

This is the principle enunciated in Art. 100.

254. Let the terms of lowest dimensions in an equation be

$$Ax^2 + By^2 + 2Cxy,$$

so that the origin is a double point. The tangents at the origin are

$$Ax^2 + By^2 + 2Cxy = 0.$$

If $C^2 > AB$, these tangents are real. If $C^2 = AB$ they coincide, and the origin is a *cusp*. If $C^2 < AB$ they are imaginary, and the origin is a *conjugate* (or isolated) point on the curve.

To determine the nature of a curve at any point we have only to transfer the origin to that point.

Ex. The curve

$$x\{x^2-(y-1)^2\}+(y-1)^4=0$$

has a treble point at $x=0$, $y=1$, since when that point is origin the equation is

$$x(x^2-y^2)+y^4=0.$$

255. *To find the circle of curvature of the curve*

$$0=ax+by+Ax^2+By^2+2Cxy+\ldots+\phi_3+\phi_4+\ldots\phi_n\ldots(1)$$

at the origin.

Let ω be the angle between the axes.

The equation

$$0=ax+by+\lambda(x^2+2xy\cos\omega+y^2)\ldots\ldots\ldots\ldots(2)$$

can by varying λ be made to represent *any* circle touching (1) at the origin.

Now all values of x and y which satisfy (1) and (2) satisfy

$$0=(A-\lambda)x^2+(B-\lambda)y^2+2(C-\lambda\cos\omega)xy$$
$$+\phi_3+\phi_4+\ldots+\phi_n\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Therefore (3) represents a curve passing through the common points of (1) and (2). Two of these common points are at O, and accordingly, O is a double point on (3). A third common point will be at O, or (3) will be (1)'s circle of curvature, if $ax+by=0$ be one of the tangents to (3) at O. That is, if $b:-a$ be one of the values of $x:y$ in

$$(A-\lambda)x^2+(B-\lambda)y^2+2(C-\lambda\cos\omega)xy=0,$$

or, if

$$\lambda=\frac{Ab^2+Ba^2-2Cab}{a^2+b^2+2ab\cos\omega},$$

then the centre and radius of the circle can be found. If $\omega=\frac{\pi}{2}$, the centre of curvature is at

$$\frac{x}{a}=\frac{y}{b}=-\frac{a^2+b^2}{2(Ab^2+Ba^2-2Cab)},$$

and the radius of curvature is

$$\frac{(a^2+b^2)^{\frac{3}{2}}}{2\,(Ab^2+Ba^2-2\,Cab)}.$$

The radius of curvature at any point of any algebraic curve is found by transferring the origin to the point*.

256. The equation for determining the length r of a line drawn in direction $[l, m]$ from the point hk to meet the curve

$$f(x, y) \equiv A_0x^n + A_1x^{n-1}y + \ldots + A_ny^n + \text{terms of lower dimensions}$$
$$\ldots + ax + by + c = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

is found by writing $h + lr$, $k + mr$ for x, y in (1). This equation is of the n^{th} degree in r. The coefficient of r^n is

$$A_0l^n + A_1l^{n-1}y + \ldots + A_nm^n,$$

or $\phi\,(l, m)$, if $\phi\,(x, y)$ denote the terms of highest dimensions in (1). The coefficients of lower powers of r may involve h, k as well as l, m. The constant term is $f(h, k)$. Thus we may write the equation for determining r

$$r^n.\,\phi\,(l, m) + r^{n-1}.\,\psi_1\,(l, m, h, k) + r^{n-2}\,\psi_2\,(l, m, h, k) + \ldots$$
$$+ r\psi_{n-1}\,(l, m, h, k) + f(h, k) = 0 \;\ldots\ldots\ldots\; (2).$$

Of course some of the coefficients ψ_1, $\psi_2 \ldots \psi_{n-1}$ may be zero.

* When the origin is transferred to the point xy of the curve $\phi\,(xy) = 0$, the equation becomes, X, Y being current co-ordinates, $\phi\,(X+x,\ Y+y) = 0$, or, since $\phi\,(xy) = 0$,

$$0 = X\frac{d\phi}{dx} + Y\frac{d\phi}{dy} + \frac{X^2}{2}.\frac{d^2\phi}{dx^2} + XY\frac{d^2\phi}{dxdy} + \frac{Y^2}{2}\,\frac{d^2\phi}{dy^2} + \text{terms of higher dimensions in}$$
X, Y.

Thus the radius of curvature of $\phi\,(xy) = 0$ at the point xy, the axes being rectangular, is

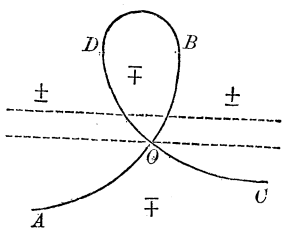
$$\frac{\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}^{\frac{3}{2}}}{\left(\frac{d\phi}{dy}\right)^2.\,\frac{d^2\phi}{dx^2} + \left(\frac{d\phi}{dx}\right)^2.\,\frac{d^2\phi}{dy^2} - 2\,\frac{d\phi}{dx}.\,\frac{d\phi}{dy}.\,\frac{d^2\phi}{dxdy}}.$$

The product of the values of r is $(-1)^n . \dfrac{f(h, k)}{\phi(l, m)}$. Thus if P, P' be two points xy, $x'y'$ from which are drawn parallel lines meeting the curve in Q_1, Q_2, ... Q_n and Q_1', Q_2', ... Q_n' respectively,

$$\frac{PQ_1 . PQ_2 \ldots PQ_n}{P'Q_1 . P'Q_2' \ldots P'Q_n'} = \frac{f(xy)}{f(x'y')}.$$

The $f(xy)$ of any point varies, therefore, as the product of the n radii drawn from the point in a given direction to meet the curve. When a point crosses the curve its $f(xy)$ changes sign (compare Arts. 176, 177).

257. Let O be a double point in a curve where the branches AOB, COD cut each other. These divide the space about



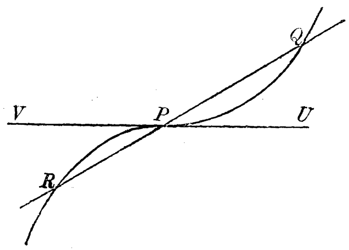
O into four compartments, AOD, DOB, BOC, COA. A point crossing the curve at O crosses either between the first and third compartments, or between the second and fourth, or tangentially.

In the first cases the point crosses the curve twice and therefore its $\phi(xy)$ does not change sign: that is, $\phi(xy)$ has the same sign in AOD as in BOC, and the same sign in DOB as in COA. The third case shews that these signs are opposite.

Thus on a given line through O the function $f(xy)$ is generally a maximum or minimum at O, and, in keeping with this, at the point O

$$\frac{df}{dx} = 0, \quad \frac{df}{dy} = 0.$$

258. A point of inflexion on a curve is a point where the curve both touches and crosses the tangent line.



If P be such a point, then of the points in which the curve is met by a line RPQ differing slightly in position from the tangent at P, two points R, Q are near to P, and move up to coincidence with P as RPQ moves into the tangential position VPU. Thus if the line $Ax + By + C$ touch the curve $\phi(xy)$ at a point of inflexion x_1y_1, three roots of the equation

$$\phi\left(x, -\frac{Ax + C}{B}\right) = 0$$

are x_1. The converse, however, is not necessarily true. Of the curves $x^3 = a^2y$, $x^3 = ay^2$, the first only has a point of inflexion at the origin.

259. If a curve of the third order have three points of inflexion, they lie on a straight line.

For let D, E, F be the three points, and BC, CA, AB the tangents at them. And let x_1y_1, x_2y_2, x_3y_3 be the co-ordinates of A, B, C, the equation to the curve being $\phi(x, y) = 0$. Then by Art. 253

$$\frac{BD \,.\, BD \,.\, BD}{CD \,.\, CD \,.\, CD} \text{ numerically} = \frac{f(x_2y_2)}{f(x_3y_3)}.$$

Similarly $\left(\frac{CE}{AE}\right)^3$, $\left(\frac{AF}{BF}\right)^3$ are numerically equal to $\frac{f(x_3y_3)}{f(x_1y_1)}$, $\frac{f(x_1y_1)}{f(x_2y_2)}$, and so $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$, and as the curve can only once cross a given side of the triangle A, B, C, the points D, E, F

lie all or one in the sides produced, and therefore D, E, F lie in a straight line (cf. Art. 217).

260. We return to equation (2) of Art. 256.

Suppose that when the line is drawn in direction $[l_1 m_1]$ two of the values of r are r_1. It does not follow that the line

$$\frac{x-h}{l_1}=\frac{y-k}{m_1}$$

touches the curve at the point $(h+l_1r_1,\ k+m_1r_1)$, for that point may be a double point.

Again, if the direction $[\lambda\mu]$ be such that $\phi(\lambda,\mu)$ is zero, one value of r is infinite. By giving to the ratio $l:m$ any one of the n values determined from the equation $\phi(l,m)=0$, we obtain a line $\dfrac{x-h}{l}=\dfrac{y-k}{m}$ meeting the curve in n points, of which one is at infinity. For instance, the lines $\phi(xy)=0$, which are all drawn through the origin, are the directions of 'the points at infinity' on the curve.

Suppose both $\phi(\lambda\mu)$ and $\psi_1(\lambda,\mu,h,k)$ to be zero. Then *two* values of r in the equation for r will be infinite, and of the n points in which the line $\dfrac{x-h}{\lambda}=\dfrac{y-k}{\mu}$ meets the curve, two will be at infinity. It does not follow that the line *touches* the curve at infinity, or in other words, that the line is an asymptote, for the curve may have a double point at infinity. For instance, let

$$f(xy)\equiv(xy-a^2)(xy-b^2),$$

so that the equation for r is

$$\begin{aligned}l^2m^2\,.\,r^4&+2lm(lk+mh)\,r^3\\&+\{(lk+mh)^2+(2hk-a^2-b^2)\,lm\}\,r^2\ldots\ldots\ldots\ldots(1),\\&+(lk+mh)(2hk-a^2-b^2)\,r+(hk-a^2)(hk-b^2)=0.\end{aligned}$$

If we write 1, 0 for l, m, that is, if we draw the line parallel to the axis of x, this equation becomes

$$\begin{aligned}0\,.\,r^4+0\,.\,r^3+k^2\,.\,r^2&+k(2hk-a^2-b^2)\,r\\&+(hk-a^2)(hk-b^2)=0\ldots\ldots\ldots\ldots(2);\end{aligned}$$

but the line $y-k=0$ is not an asymptote. For the curve is not a *proper* curve of the 4th degree, but consists of the two hyperbolas $xy=a^2$, $xy=b^2$, which being co-asymptotic have double contact at infinity. The points of contact are double points on the compound curve, and *every* line parallel to a co-ordinate axis passes through one or other of these points. In order to get a tangent to the compound curve at one of these points it is necessary that in (1) *three* values of r should be infinite; that is, k must be zero: or, the line $y=0$ is an asymptote.

If we look upon h, k as 'current' co-ordinates, we get the equation to this asymptote by equating to zero the coefficient of r^2 in (2).

Similarly $x=0$ is an asymptote.

All possible *directions* of asymptotes of a curve are comprised in the equation $\phi(l, m)=0$. Let (λ, μ) be a direction hence chosen.

In the equation to the curve write $x+\lambda r$, $y+\mu r$ for x, y, and in the result equate to zero the coefficient of the highest visible power of r.

This last result is the equation to the asymptote or asymptotes in direction $[\lambda\mu]$.

Ex. (1) $x^3+y^3+3axy=0$.

Here $\phi(l, m)\equiv l^3+m^3=0$ if $l+m=0$. *Then* the equation for r is

$$(x+\lambda r)^3+(y+\mu r)^3+3a(x+\lambda r)(y+\mu r)$$
$$\equiv 3r^2(x\lambda^2+y\mu^2+a\lambda\mu)+\ldots=0.$$

Since $\lambda+\mu=0$, $\lambda^2=\mu^2=-\lambda\mu$. Thus the asymptote is

$$x+y=a.$$

(2) $x^3-y^2x-by^2-a^3=0$.

Here $$\phi(l, m)\equiv l^3-lm^2,$$

$$\psi_1(l, m, x, y)\equiv 3l^2x-m^2x-2lmy-bm^2.$$

If we suppose

$$\lambda^2 - \mu^2 = 0,$$

$$\psi_1(\lambda, \mu, x, y) = x \pm y - \frac{b}{2}.$$

Thus $$x \pm y - \frac{b}{2} = 0$$

are asymptotes in direction

$$\lambda + \mu = 0, \quad \lambda - \mu = 0.$$

If $\lambda = 0$, then

$$\psi_1(\lambda, \mu, x, y) = -x - b;$$

therefore $x + b = 0$ is an asymptote.

If in the equation $f(xy) = 0$, y becomes infinite when x has the value a, then $x = b$ is an asymptote. Similarly if when y has the value b, k is infinite, $y = b$ is an asymptote. We leave the proof to the reader.

Thus the curve

$$y = \frac{x^2}{x - a}$$

has the line $x = a$ for an asymptote.

261. A curve whose equation is of the third degree is called a cubic.

The general equation of the third degree contains 10 terms. If we have given 9 points on a cubic we have therefore 9 homogeneous linear equations for determining the 9 ratios of the constants. Such a system of equations, if they be independent, admits of one and only one solution: in other words, a cubic is, in general, *uniquely* determined by 9 points.

The general equation of the n^{th} degree contains $\frac{(n+1)(n+2)}{2}$ terms, and thus a curve of the n^{th} degree is in general uniquely determined by $\frac{n(n+3)}{2}$ points.

The datum of one point enables us to eliminate one of the

arbitrary constants in a curve's equation. And any datum may, by counting the number of constants it enables us to eliminate, be reduced to the form of points. Thus the focus of a conic is equivalent to two points, for the general equation to a conic with focus at a given point (a, b) is

$$(x-a)^2+(y-b)^2=(Ax+By+C)^2,$$

involving three constants. The centre of a conic is equivalent to two points, for when the given centre is origin the equation involves only three independent constants.

262. Since any two cubics have 9 common points, it is clear that these 9 points are not sufficient to determine a cubic uniquely, for the cubic in question may be *either* of the two cubics aforesaid. Nor indeed do they determine a finite number of cubics, for if

$$U=0 \quad\text{.............................}\quad (1),$$

$$V=0 \quad\text{.............................}\quad (2),$$

be the first two cubics, the *infinity* of cubics obtained by giving different constant values to λ in

$$U+\lambda V=0 \quad\text{..........................}\quad (3),$$

pass through the same nine points.

Suppose 8 points given on a cubic. Draw any two cubics through these 8 points and let P be their 9th common point. Then as an infinity of cubics can be drawn through the nine points it follows that of the cubics through the 8 given points an infinite number pass through a 9th point which can be predicted. It does not *immediately* follow that they *all* pass through this 9th point, though that seems to be true.

If the two cubics through the 8 given points be improper and have a common part, P may be *any* ninth common point.

263. We conclude this chapter by tracing the curve

$$y^2(x-4a)=ax(x-3a)$$

from its equation.

We suppose a positive. Writing the equation in the form

$$y^2 = \frac{ax(x-3a)}{x-4a},$$

we see that when $x = 4a$, y is infinite. Thus $x = 4a$ is an asymptote. It will be found that this is the only asymptote.

The curve is symmetrical with respect to the axis of x, for the equation involves no odd power of y.

The curve passes through the origin, and the line $x = 0$ is a tangent at the origin (Art. 100).

y^2 is negative when x is negative, positive from $x = 0$ to $x = 3a$, negative from $x = 3a$ to $x = 4a$, and positive from $x = 4a$ to $x = \infty$. Thus the curve lies partly between the lines $x = 0$, $x = 3a$, and partly on the positive side of the line $x = 4a$.

Approximately, when x is very great,

$$y^2 = ax\left(1 - \frac{3a}{x}\right)\left(1 + \frac{4a}{x}\right) = ax + a^2.$$

Thus $y^2 = ax + a^2$ is a *parabolic* asymptote.

The y^2 of the curve is greater than that of the parabolic asymptote. This determines the side of the asymptote on which the curve lies.

The line $x = 3a$ is a tangent at the point $x = 3a$. To prove this we may transfer the origin to that point, or simply observe that corresponding to this value of x two values of y are equal. The third value is infinite.

The branch between $x = 0$ and $x = 3a$ is an oval. The rest of the curve consists of two infinite branches.

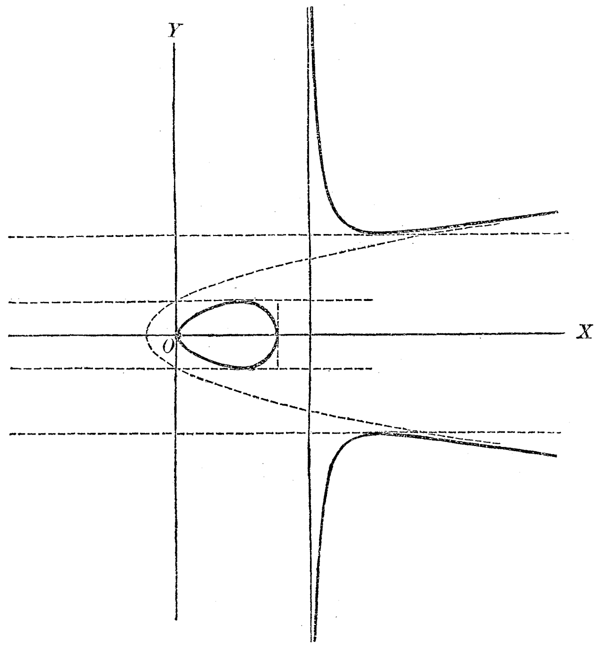
Solving the equation with respect to x we have

$$x = \frac{3a^2 + y^2 \pm \sqrt{(y^2 - 9a^2)(y^2 - a^2)}}{2a}.$$

Thus the lines $y^2 = 9a^2$, $y^2 = a^2$ are tangents to the curve.

The first pair are an interior limit of the infinite branches; the second pair bound the oval.

This example is taken from Hymers' *Conic Sections*, from which also have been selected some of the examples at the end of this chapter.



EXAMPLES ON CHAPTER XIV.

1. A curve of the third degree cannot have a double tangent, nor two double points, nor two ovals, nor an oval and a double point.

2. A curve of the fourth degree cannot have four double points, nor four ovals, nor, together with one, two, three double points, ovals three, two, one. (See Art. 184).

3. The circle

$$x^2 + y^2 - 10ax + 4ay - 3a^2 = 0$$

has contact of the 2nd order with the curve $y^2 = 4ax$ at the point $(a, 2a)$.

4. Find the equation to the parabola which has contact of the third order with the conic

$$x^2 + 2axy + by^2 + 2cy = 0$$

at the origin. (Salmon's *Conic Sections*, Art. 244.)

5. If a, b, c be in ascending order of magnitude, the curve

$$y^2 = (x-a)(x-b)(x-c)$$

consists of an oval and an infinite branch.

What becomes of the oval (1) when b decreases to equality with a, and (2) when b increases to equality with c? (Salmon's *Higher Plane Curves*, Art. 32.)

6. The curve

$$x^3 - y^3 - 3x^2 - 3y^2 - xy + 2x - 2y + 1 = 0$$

has a double point at $(1, -1)$.

7. The curves

$$y^5 = ax^2, \quad y^3 = ax^2,$$

have each a cusp at the origin, and the radii of curvature at that point are respectively infinitely great and infinitely small.

8. Prove that if O, Q be two points on a curve, and QR a perpendicular to the tangent at O, then the limiting value of $\frac{QO^2}{QR}$ when Q moves up to and coincides with O is the diameter of curvature at O.

9. Find the radius of curvature at any point $x'y'$ of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and prove that if W be any point on the diameter DD', the chord of curvature at P in direction PW is $\frac{2CD^2}{PW}$.

10. The locus of the centre of curvature at any point of the parabola $y^2 = 4ax$ is the 'semicubical parabola'

$$27ay^2 = 4(x-2a)^3.$$

Where does this cut the parabola?

11. The sides A_1A_2, $A_2A_3 \dots A_mA_1$ are met by a curve of the n^{th} degree. If $P(A_{12})$ denote the product of all the n segments of the side A_1A_2 that are measured from A_1, &c., then numerically

$$P(A_{12}) \,.\, P(A_{23}) \dots P(A_{n1}) = P(A_{1n}) \dots P(A_{32}) \,.\, P(A_{21}).$$

(Carnot's Theorem.)

12. The origin and the points where $x = \pm a\sqrt{3}$ are points of inflexion on the curve

$$x^3 = (a^2 + x^2)\, y.$$

13. Find the asymptotes of the curves

(1) $y^2(x-2a) = x^3 - a^3$,

(2) $xy^2 + x^2y = a$,

(3) $4x^3 = (x+3a)(x^2+y^2)$.

14. A curve of the n^{th} degree cannot have more than n asymptotes.

If the sides of the triangle of reference be asymptotes of a cubic, the equation to the cubic is of the form

$$\alpha\beta\gamma + l\alpha + m\beta + n\gamma = 0\,;$$

and the three points where the curve cuts its asymptotes lie in a straight line.

15. Prove that where x is very great the curve

$$y^2 = \frac{x^3 - a^3}{x - 2a}$$

nearly coincides with

$$\pm \frac{y}{x} = 1 + \frac{a}{x} + \frac{3a^2}{2x^2},$$

and hence find the asymptotes and trace the curve.

16. Prove by solving the equation with respect to y^2 that the curve

$$(y^2 - x^2)(x - 1)\left(x - \frac{3}{2}\right) = 2\{y^2 + x(x-2)\}^2$$

is bounded by the lines

$$x = -\frac{1}{10}, \quad x = \frac{3}{2}.$$

Prove also that it has three double points and four double tangents.

17. On the radius vector of a straight line are taken two points at the same given distance from the radius vector's extremity. Prove that they lie on the same algebraic curve, to which the given line is an asymptote.

18. From a given point O a line is drawn meeting a given curve of the n^{th} degree in $P_1, P_2, \ldots P_n$. In this line Q is taken, such that

$$\frac{1}{OQ} = \frac{1}{OP_1} + \frac{1}{OP_2} + \cdots + \frac{1}{OP_n}.$$

Prove that the locus of Q is a straight line.

19. If a line be drawn in a given direction in the plane of an algebraic curve the locus of the centre of mean position of the points in which the line meets the curve is a straight line.

N.B. The centre of mean position of the points x_1y_1, $x_2y_2 \ldots x_ny_n$ is the point

$$x = \frac{x_1 + x_2 + \ldots + x_n}{n}, \quad y = \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

20. The corner of the leaf of a book is doubled down so as to form a triangle of constant area. Shew that the locus of the corner is the lemniscate

$$r^2 = a^2 \sin 2\theta.$$

21. Assuming that all cubics passing through 8 given points pass also through a 9th, deduce Pascal's Theorem.

22. Trace the following curves:

(1) The four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$,

(2) $x^3 + y^3 = a^3$,

(3) The cardioid $r = a(1 - \cos\theta)$,

(4) The conchoid $r = a + b \operatorname{cosec}\theta$,

(5) $\left(\tan\dfrac{x}{a}\right)^2 + \left(\tan\dfrac{y}{b}\right)^2 = 1$,

and prove that the curve

$$r = a + \frac{b}{\theta}$$

has a circular asymptote.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

1. (1) $r=\sqrt{2},\ \theta=\frac{\pi}{4}$. (2) $r=\sqrt{5},\ \theta=\pi+\tan^{-1}2$.

(3) $r=-3,\ \theta=0$. (4) $r=4,\ \theta=-\frac{\pi}{2}$ (cf. Ex. 12).

2. (1) $x=\sqrt{3},\ y=1$. (2) $x=1,\ y=0$. (3) $x=-\frac{\sqrt{3}}{2},\ y=\frac{1}{2}$.
(4) $x=-1,\ y=0$.

3. $\sqrt{185}$; $2\sqrt{2}+\sqrt{17}+\sqrt{13}$. 4. $\frac{5}{2}$.

5. For D, $x=\frac{x_2+x_3}{2},\ y=\frac{y_2+y_3}{2}$. For the middle point of EF, $x=\frac{2x_1+x_2+x_3}{4},\ y=\frac{2y_1+y_2+y_3}{4}$. 6. $\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}$ (symmetrical expressions). 8. $\frac{c}{2},\ \frac{b}{2}$; $\frac{\sqrt{b^2+c^2+2bc\cos A}}{2}$.

11. Q moves from B to an ∞ and from an ∞ to A.

12. The point $\{(-1)^n r,\ n\pi+\theta\}$, n being any positive or negative integer, or zero.

CHAPTER II.

1. It can be reduced to $(x^2+y^2-a^2)^3+27x^2y^2a^2=0$.

2. $\frac{5}{2},\ -\frac{5}{3}$; $-\frac{C}{A},\ -\frac{C}{B}$. 3. $\frac{3\pi}{4},\ \tan^{-1}\left(-\frac{p}{q}\right),\ \frac{5\pi}{6}$.

4. $-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}$; $-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}$; $-\frac{\sqrt{3}}{2},\ \frac{1}{2}$;
$\frac{1}{2},\ \frac{\sqrt{3}}{2}$; $-\frac{g}{\sqrt{f^2+g^2}},\ \frac{f}{\sqrt{f^2+g^2}}$; $\frac{g}{\sqrt{f^2+g^2}},\ \frac{f}{\sqrt{f^2+g^2}}$,
$-\frac{B}{\sqrt{A^2+B^2}},\ \frac{A}{\sqrt{A^2+B^2}}$, ($f$ and A are supposed positive).

7. $\frac{\omega}{2}$; $\frac{\pi}{2},\ \tan^{-1}\frac{A\sin\omega}{A\cos\omega-B}$.

8. (Ex. 5). $\frac{1}{\sqrt{5+2\sqrt{2}}}$, $\frac{2}{\sqrt{5+2\sqrt{2}}}$; $\frac{l}{B} = \frac{m}{-A}$

$$= \frac{1}{\sqrt{A^2+B^2-2AB\cos\omega}}.$$

9. $r(\cos\theta - \sin\theta) = 3$; $\sqrt{3}\,.\,x - y + 10 = 0$.

10. The point lies on the line.

CHAPTER III.

1. $\left(-\frac{1}{38}, \frac{31}{38}\right)$. 2. The point is $(-7, -3)$.

4. $x - y = 3$, $x + y = -1$. 5. $y - k = m(x - h)$. 6. $x = y$.

7. (1) $\frac{x}{a} = \frac{y}{b}$. (2) $x + 5y + 3 = 0$. (3) $x = 4$. (4) $y = 0$.

8. $\frac{6}{\sqrt{2}}$, $\frac{6}{\sqrt{2}}$; $\frac{12}{\sqrt{2}}$. 9. $\tan^{-1}\frac{1}{7}$; 0, $\frac{7}{\sqrt{10}}$.

11. $2x - 3y = 3$, $3x + 2y + 2 = 0$.

12. $(a-c)x + (b-d)y = 0$; $\sqrt{\frac{(ad-bc)^2}{(a-c)^2+(b-d)^2}}$.

14. $\tan^{-1} 3\sqrt{3}$; $5x - 7y = 12$. 17. $x - y = 0$, $x + y = 0$.

19. $x^2 + y^2 = c^2$. 20. $xy = -\frac{a^2}{2}$.

21. $(x^2+y^2)^2 = a^2(x^2-y^2)$.

23. The expressions substituted for x, y are *always* linear.

24. $3y^2 = 2l(y + 2x)$.

25. $x = x'\cos\alpha + y'\cos\beta$, $y = x'\sin\alpha + y'\sin\beta$ (Art. 26).

26. $\frac{(am+c)^2}{2m(m+1)}$. 27. Art. 34; equal and opposite inclinations to OY. 28. $x + y\cos\omega = h + k\cos\omega$.

29. Take AB, AC for axes. If $AB = c$ and $AC = b$, the equations to FC, KB, AL are then

$$x\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{y}{b}=1, \quad \frac{x}{c}+y\left(\frac{1}{b}+\frac{1}{c}\right)=1, \quad \frac{x}{b}=\frac{y}{c}.$$

33. If $Al + Bm = \frac{1}{r}$ be the type of the given lines, then the locus is the straight line $\frac{1}{r} = \Sigma(A)\,.\,l + \Sigma(B)\,.\,m$.

34. Cf. Art. 36. 35. $11x - 22y + 25 = 0$.

36. If x_1y_1, x_2y_2 be opposite corners of a parallelogram whose sides are parallel to the axes, the other corners are x_1y_2, x_2y_1.

37. $y = 3$, $y + x\sqrt{3} = 3 + \sqrt{3}$.

38. A case of Chap. v. Ex. 23. AL, BM, CN meet in

$$\frac{y}{fgh} = \frac{\frac{x}{f} + \frac{y}{h} - 1}{fg - h^2} = \frac{\frac{x}{-g} + \frac{y}{h} - 1}{fg - h^2}.$$

The straight line is $\dfrac{y}{fgh} + \dfrac{\frac{x}{f} + \frac{y}{h} - 1}{fg - h^2} + \dfrac{\frac{x}{-g} + \frac{y}{h} - 1}{fg - h^2} = 0$.

39. Ex. 34.

CHAPTER IV.

1. (1) The lines $x - a$, $x + a$.

(2) The lines $x - a + \sqrt{-1}\,(y - b)$, $x - a - \sqrt{-1}\,(y - b)$.

(3) The lines $x - a$, $y - b$. (4) The lines $y + 3x$, $y + x$.

(5) Two coincident lines $y + 2x$.

(6) The lines $x = a$, $x = a\dfrac{-1 + \sqrt{-3}}{2}$, $x = a\dfrac{-1 - \sqrt{-3}}{2}$.

2. $(ac' - a'c)^2 = 4\,(ab' - a'b)\,(bc' - b'c)$. Art. 32, (1).

5. They bisect the lines joining the point (lr, mr) to the points $(l'r, m'r)$, $(-l'r, -m'r)$.

6. $\dfrac{Ax + By + C}{A'x + B'y + C'} = \pm\sqrt{\dfrac{A^2 + B^2}{A'^2 + B'^2}}$.

8. It is given that $abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2 = 0$.

9. Yes, two coincident straight lines. 10. $\dfrac{11}{5}$.

11. f and g cannot be so determined. (Ex. 7 & Art. 50.)

12. $\dfrac{(C - C')(c - c')}{Ab - Ba}$.

13. It represents the lines $\theta = 0$, $\theta = \dfrac{\pi}{3}$, $\theta = \dfrac{2\pi}{3}$.

14. Arts. 12, 38, Euclid vi. 3.

15. $Ax_1 + By_1 + C : Ax_2 + By_2 + C$.

17. By Ex. 16, four straight lines through the intersection of

the given lines (*four*, if the division may be either internal or external and the order of comparing the segments be unfixed).

19. The locus is a *finite* straight line: areas are not subject to sign. 20. A straight line.

CHAPTER V.

2. $\frac{2\Delta}{3a}, \frac{2\Delta}{3b}, \frac{2\Delta}{3c}$.

3. $l\alpha + m\beta + n\gamma = lf + mg + nh$; $(am - bl)\beta + (an - cl)\gamma = 0$.

4. $-a\alpha + b\beta + c\gamma = 0$; $b\beta + c\gamma = 0$.

5. The line is parallel to $\beta \cos B - \gamma \cos C$.

6. $l(m'n'' - m''n') + l'(m''n - mn'') + l''(mn' - m'n) = 0$.

8. $\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) + \alpha_2(\beta_3\gamma_1 - \beta_1\gamma_3) + \alpha_3(\beta_2\gamma_1 - \beta_1\gamma_2) = 0$.

9. $\tan^{-1}\frac{(mn'-m'n)\sin A+(nl'-n'l)\sin B+(lm'-l'm)\sin C}{ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B-(lm'+l'm)\cos C}$ (by Cartesian co-ordinates).

10. $\beta + \gamma - \alpha = 0$, $\gamma + \alpha - \beta = 0$, $\alpha + \beta - \gamma = 0$.

11. $\frac{n_2\alpha_1 + n_1\alpha_2}{n_1 + n_2}, \frac{n_2\beta_1 + n_1\beta_2}{n_1 + n_2}, \frac{n_2\gamma_1 + n_1\gamma_2}{n_1 + n_2}$, if $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ be the given points and $n_1 : n_2$ the given ratio.

13. $\frac{\alpha - \frac{2\Delta}{19f}}{10} = \frac{\beta - \frac{2\Delta}{19f}}{-9} = \gamma - \frac{4\Delta}{19f}$.

14. $\frac{l}{a} : \frac{m}{b}$. 16. $mv + nw = 0$, $nw + lu = 0$, $lu + mv = 0$.

17. AL, BM, CN are $mv + nw$, $nw + lu$, $lu + mv$; OA, OL are $mv - nw$, $2lu + mv + nw$.

18. Ex. 14.

19. $AM = \frac{acl}{cl - an}$, $AN = \frac{abl}{bl - am}$; distance from A

$$= \frac{\sin A}{\sqrt{\frac{1}{AM^2} + \frac{1}{AN^2} + \frac{2\cos A}{AM \cdot AN}}}.$$

20. LAD may be made the triangle of reference and N assumed to be the point $l\alpha = m\beta = n\gamma$.

21. $mv + nw$, $nw + lu$, $lu + mv$ are AC, LM, BD. Let LM meet AC, BD in R, T. Then $mv - nw$, $nw - lu$, $lu - mv$ are CT, LN, BR, which therefore meet in a point P: $lu - 2mv + nw$, $2lu - mv + nw$, $lu - mv + 2nw$ are PM, RU, TW, if U, W be the intersections of BL, CT; CL, BR.

22. If P be a point fgh, then the co-ordinates of any point A' in AP are proportional to $f + \frac{2\Delta}{a} . \lambda$, g, h, if $\lambda = PA' : A'A$, &c. (Ex. 11, and Art. 73).

23. Let AL, BM, CN meet in $l\alpha = m\beta = n\gamma$; the straight line is $l\alpha + m\beta + n\gamma$.

CHAPTER VI.

1. $\left(-\frac{15}{2}, 0\right)$; $\frac{1}{2}\sqrt{105}$.

2. $x^2 + y^2 - (x' + x'')x - (y' + y'')y + x'x'' + y'y'' = 0$.

4. $\pi - \alpha$ if $\alpha > \pi$ and < 0.

5. $60°$; $\left(\frac{2h-k}{3}, \frac{2k-h}{3}\right)$; $\sqrt{\frac{h^2 + hk + k^2}{3}}$.

6. $-\frac{A \pm \sqrt{A^2 - 4C}}{2}$, $-\frac{B \pm \sqrt{B^2 - 4C}}{2}$. 8. $A^2 = B^2 = 4C$.

9. $\frac{x-a}{x'-a} = \frac{y-b}{y'-b}$. 10. Cf. Euclid III. 21.

12. $x \cos \alpha + y \cos \alpha = c$. 13. $(Aa + Bb + C)^2 = (A^2 + B^2)c^2$.

14. Yes, at the origin.

15. Determine n so that $y = mx + n$ may touch the circle.

16. Take $x^2 + y^2 = c^2$ for the circle and $y = 0$ for the diameter.

17. $xx' + yy' = 2a(x + x')$. 18. $xx' + yy' + 2\cos\omega(xy' + yx') = c^2$.

19. A circle concentric with the square.

20. AB is divided harmonically.

23. $x - 2y = 9$; $(9, 9)$. 24. $a = r\cos(\theta - \alpha)$.

25. The line joining the centre to the fixed point is a diameter.

27. Use equation (2) of Art. 93. The locus is $\frac{1}{r} = \frac{a\cos\theta}{a^2 - c^2}$.

29. A circle: use equation (2) of Art. 93.

30. A circle. The circle becomes a straight line.

31. A circle through O: the diameter through O is perpendicular to the line.

CHAPTER VII.

1. Without. 2. The distance between the centres is the sum or difference of the radii.

3. $(\alpha' A - \alpha A')^2 + (\alpha' B - \alpha B')^2 = \{\alpha' \sqrt{A^2 + B^2 - 4C} \pm \alpha \sqrt{A'^2 + B'^2 - 4C'}\}^2$.

4. If $AB = c$ and $AC = b$, the equation is

$$x^2 + y^2 + 2xy \cos A = cx + by.$$

5. Assume for the equation $x^2 + y^2 + Ax + By + C = 0$. There are three equations for determining A, B, C.

6. $Ax + By + 2C = 0$. 7. (1) $a(y + y') = 2xx'$.

(2) $a^2(y + 2y') = 3x'^2 x$. (3) $\dfrac{y - y'}{a} = \cos \dfrac{x'}{b} \cdot \dfrac{x - x'}{b}$.

(4) $\dfrac{x}{\sqrt[3]{x'}} + \dfrac{y}{\sqrt[3]{y'}} = a^{\frac{2}{3}}$. 11. $a = b = \dfrac{c'}{\cos \omega}$.

13. The equation for determining the given ratio $l : l'$ is

$$(x'^2 + y'^2 - c^2)\, l^2 + (x^2 + y^2 - c^2)\, l'^2 + 2(xx' + yy' - c^2)\, ll' = 0.$$

Compare Art. 179.

14. "and prove that..." : first prove that L is the pole of MN; then similarly M is the pole of NL; therefore (Art. 101) N is the pole of LM.

CHAPTER VIII.

1. $x^2 + y^2 = \dfrac{1}{2}(x + y - 1)^2$.

2. $y - x = a$, $y + x = a$; $x + y = 3a$, $x - y = 3a$.

4. The equation is $x = y$. 5. 90°; 90°.

10. The focal distance of the point $(4, -2)$ is $\dfrac{17}{4}$: the normal is $4x - y = 18$. 11. $\left(-\dfrac{9}{4}, \dfrac{3\sqrt{3}}{2}\right)$, $\left(-\dfrac{9}{4}, -\dfrac{3\sqrt{3}}{2}\right)$.

14. The equation for the ordinate is $y^2 - \dfrac{4a}{m} y + \dfrac{4ac}{m} = 0$.

15. Within. 16. The vertices are $(-2, 3)$, $(2, -3)$; the foci are $(-1, 3)$, $(1, -3)$. 17. $y^2 = 6x$.

18. $\sqrt{x} + \sqrt{y} = \sqrt{2a\sqrt{2}}$. 21. The ratio, $l : l'$ is found from $(y'^2 - 4ax')\, l^2 + (y^2 - 4ax)\, l'^2 + 2ll'(yy' - 2a\,\overline{x + x'}) = 0$.

22. $r = \dfrac{4a \cos \theta}{\sin^2 \theta}$. 23. $16a^2$.

24. *One* form is $r \sin^2 \theta = 4a (\sin \theta + \cos \theta)$. 26, 27. The m of the normal from $x'y'$ is found from $y' = mx' - 2am - am^3$. The values of m are $-\dfrac{y_1}{2a}$, $-\dfrac{y_2}{2a}$, $-\dfrac{y_3}{2a}$.

28. Eliminate x', y' from $\dfrac{x}{x'} = \dfrac{y}{y'}$, $y'k = 2a(x' + h)$, $y'^2 = 4ax'$.

29. The parabola $y^2 = 2a(x - a)$. 30. Use polar co-ordinates.

32. When h is indefinitely increased, the curve tends to become two parallel lines.

CHAPTER IX.

1. $58 \{(x-1)^2 + (y+2)^2\} = \dfrac{1}{9}(7y - 3x + 2)^2$. 2. $\sqrt{l^2 + m^2}$.

3. $\dfrac{1}{\sqrt{2}}$, $\sqrt{\dfrac{2}{5}}$; $\left(\pm \dfrac{1}{\sqrt{2}}, 0\right)$, $\left(0, \pm \dfrac{\sqrt{22}}{5}\right)$.

5. $ex \pm y = a$, $ex \pm y = -a$; at E, E'. 6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

7. $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} = \dfrac{x + x'}{a}$, $\dfrac{a^2x}{x' - a} - \dfrac{b^2y}{y'} = \dfrac{a^2x'}{x' - a} - b^2$.

9. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1$, $ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$.

11. Without. 12. $64x^2 = 49y^2 = \dfrac{56}{5}$.

15. $4(a^2y^2 + b^2x^2 - a^2b^2) = (x^2 + y^2 - a^2 - b^2)^2 . \tan^2\alpha$, if α be the given angle.

16. Four from points in SS'; two from other points in the major axis: from a point G' in the minor axis four or two according as $CG' <$ or $> \dfrac{a^2e^2}{b}$ (Art. 126).

21. When $CP = CD$.

25. The locus of CP's middle point is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{1}{4}$.

26. $(a^2 + b^2)(yx' - xy') = (a^2 - b^2)(xx' - yy' - x^2 + y^2)$.

29. SP, $S'P$ are bisected by CY, CY'.

30. $-\dfrac{mca^2}{a^2m^2 + b^2}$, $\dfrac{b^2c}{a^2m^2 + b^2}$; the straight line $y = -\dfrac{b^2}{ma^2} . x$.

31. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{1}{2}$. (Art. 126).

33. One pair bisects the chords joining the extremities of the other two. (Arts. 67, 140). If the diameters be $y = mx$, $y = -\dfrac{b^2x}{a^2m}$, $y = m'x$, $y = -\dfrac{b^2x}{a^2m'}$, the condition is $m' - m = \pm\left(\dfrac{a}{b}\cdot mm' + \dfrac{b}{a}\right)$.

34. $\dfrac{p}{r} = \sin\theta \pm \sqrt{e^2\sin^2\theta - \cos^2\theta}$.

37. $x^2\left(\dfrac{1}{c^2} - \dfrac{1}{a^2}\right) - y^2\left(\dfrac{1}{b^2} - \dfrac{1}{c^2}\right) = 0$. 38. The four tangents are

$$x\sqrt{c^2 - b^2} \pm y\sqrt{a^2 - c^2} = \pm c\sqrt{a^2 - b^2}. \quad \text{(Art. 128).}$$

39. $\sec^{-1}\dfrac{c}{ab}\sqrt{a^2 + b^2 - c^2}$; $\dfrac{1}{c}\sqrt{(a^2 - c^2)(c^2 - b^2)}$. (See Ex. 23).

41. An ellipse: transfer the origin to the point and use polar co-ordinates. 43. A circle. (Art. 129).

44. (Art. 127). The foci are angular points of a rectangle whose sides are perpendicular to the tangents.

45. Transform the equation by moving the origin to the vertex.

46. $\dfrac{l}{a + r} = 1 + e\cos\theta$ (S being pole);

$r^2(a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta) = (a^2 - b^2)^2$ (C being pole), (cf. Art. 127);

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = \frac{a^4}{x^4}.$$

47. Take A, A' successively for poles.

48. A rectilinear ellipse SS'. 49. The circle on AA'.

51. A line parallel to the tangent at the given point.

52. $(Aa)^2 + (Bb)^2 = C^2$. 53. Ex. 52.

54. Transfer the origin to the point.

CHAPTER X.

1. $13(x^2 + y^2) = 2(x - 5y + 3)^2$. 2. $\sqrt{6}$; $\dfrac{6\sqrt{3}}{5}$, $6\sqrt{\dfrac{3}{5}}$.

3. $2\sqrt{\dfrac{2}{5}}$; $\dfrac{\sqrt{A^2 + B^2}}{A}$ or $\dfrac{\sqrt{A^2 + B^2}}{B}$, according as C is positive or negative. 4. $\left(-\dfrac{1}{10}, -\dfrac{7}{4}\right)$; $\dfrac{1}{2}\sqrt{\dfrac{243}{5}}$, $\dfrac{1}{5}\sqrt{\dfrac{243}{2}}$.

8. Salmon's *Conic Sections*, 4th Ed., Arts. 185, 202.

9. $\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$. 10. At the points where $x = 8a$.

13. (1) $x = 2a$; $y = \pm(x+a)$. (2) $xy^2 + x^2y = 0$.

(3) $x \pm \sqrt{3} . y = \dfrac{2a}{3}$. 15. $x + a = \pm y$.

16. The double points are (0, 0), (1, 1), (1, −1).

Two of the double tangents are $x = -\dfrac{1}{10}$, $x = \dfrac{3}{2}$.

22. (4) See Ex. 17. (5) This curve is an endless pattern.

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